Evoluzione della varianza della posizione nel tempo nel pacchetto gaussiano usando le metodo di Heisenberg

(1)

Equazioni del moto per gli operatori

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}}{m} \qquad \frac{d\hat{p}}{dt} = 0$$

soluzione

$$\begin{cases} \hat{x}(t) = \hat{x}(0) + \frac{\hat{p}(0)}{m} t \\ \hat{p}(t) = \hat{p}(0) \end{cases}$$

p conservato

calcoliamo prima

$$\langle \hat{p}^2(t) \rangle = \langle p^2(0) \rangle$$

$$\langle \hat{p}(t) \rangle = \langle \hat{p}(0) \rangle = \frac{\hbar k_0}{m}$$

$$\langle \hat{p}^2(t) \rangle - \langle p(t) \rangle^2 = \langle \hat{p}^2(0) \rangle - \langle p(0) \rangle^2 = \sigma_p = \frac{\hbar^2}{4\sigma_x^2}$$

indipendente dal tempo p conservato

$$\langle \hat{x}^2(t) \rangle = \left\langle \left(\hat{x}(0) + \frac{\hat{p}(0)}{m} t \right)^2 \right\rangle =$$

$$= \left\langle x^2(0) + \frac{\hat{p}^2(0)}{m^2} t^2 + \frac{t}{m}(xp + px) \right\rangle$$

$$\langle x(t) \rangle = \langle x(0) \rangle + \frac{\langle p(0) \rangle}{m} t$$

$$\langle \Delta x^2(t) \rangle = \langle x^2(t) \rangle - \langle x(t) \rangle^2 =$$

$$= \langle x^2(0) \rangle - \langle x(0) \rangle^2 + \frac{t^2}{m^2}\left(\langle p^2 \rangle - \langle p \rangle^2\right) +$$

$$+ \frac{t}{m}\langle x p + p x \rangle - 2\frac{t}{m}\langle x \rangle \langle p \rangle$$

$$\langle x p + p x \rangle = \langle x p + (x p)^{\dagger} \rangle = 2 \mathrm{Re} \langle x p \rangle$$

$$\langle x p \rangle = \int dx\, \psi^*(x)\, x \left(-i\hbar \frac{\partial}{\partial x}\right) \psi(x)$$

$$\psi(x) = \psi_0(x - x_0)\, e^{i k_0 x}$$ radice della funzione gaussiana

$$\psi'(x) = \psi_0'(x - x_0)\, e^{i k_0 x} + i k_0 \psi(x)$$

$$\langle x p \rangle = \int dx\, \psi_0\, x \left[\overset{-i\hbar}{\psi_0'} + \hbar k_0 \psi_0\right]$$

$$2 \mathrm{Re} \langle x p \rangle = 2 \mathrm{Re}\left[-i\hbar \int dx\, \psi_0(x - x_0)\, x\, \psi_0'(x - x_0) + \right.$$

$$\left. + \left(\int dx\, \psi_0(x - x_0)\, x\, \psi_0(x - x_0)\right) \hbar k_0\right]$$

$$2 \mathrm{Re} \langle x p \rangle = 2\hbar k_0 \langle x_0 \rangle$$

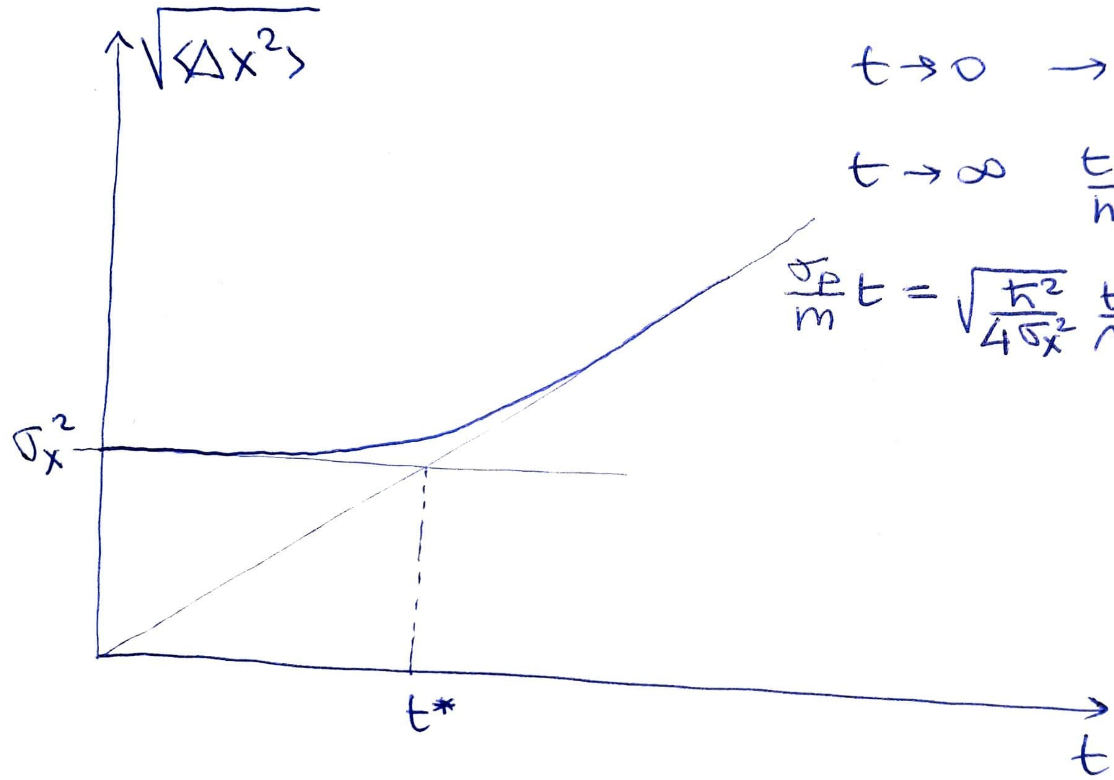
$$\langle \Delta x^2(t) \rangle = \langle \Delta x^2(0) \rangle + \frac{t^2}{m^2}\sigma_p^2 + \cancel{\frac{t}{m} 2\hbar k_0 x_0 - 2\frac{t}{m} x_0 \hbar k_0}$$

$$\langle \Delta x^2(t) \rangle = \sigma_x^2 + \frac{t^2}{m^2}\sigma_p^2 = \sigma_x^2\left(1 + \frac{t^2}{m^2}\frac{\sigma_p^2}{\sigma_x^2}\right)$$

$$\langle \Delta x^2(t) \rangle = \sigma_x^2 \left(1 + \frac{t^2}{m^2} \frac{\hbar^2}{4\sigma_x^4} \right)$$

$$\boxed{\langle \Delta x^2(t) \rangle = \sigma_x^2 \left(1 + \left(\frac{\hbar t}{2m\sigma_x^2} \right)^2 \right)}$$

$$\langle \Delta x^2(t) \rangle = \sigma_x^2 \left(1 + \frac{t^2}{m^2} \frac{\sigma_p^2}{\sigma_x^2} \right)$$



$t \to 0 \quad \to \sigma_x^2$

$t \to \infty \quad \frac{t^2}{m^2}\sigma_p^2$

$$\frac{\sigma_p}{m} t = \sqrt{\frac{\hbar^2}{4\sigma_x^2}} \frac{t}{m} = \frac{\hbar t}{2m\sigma_x}$$

t^* : $\quad \frac{t^2}{m^2}\sigma_p^2 = \sigma_x^2 \qquad t^* = \frac{m\sigma_x}{\sigma_p} = \frac{m\sigma_x}{\sqrt{\frac{\hbar^2}{4\sigma_x^2}}}$

$$\boxed{t^* = \frac{\sqrt{2}\, m \sigma_x^2}{\hbar}}$$

confrontare con la legge della diffusione (a d=1)

$$\langle \Delta x^2(t) \rangle = 2Dt \Rightarrow \sqrt{\langle \Delta x^2(t) \rangle} = \sqrt{2Dt}$$