Oscillatore Armonico

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

$$x = \frac{\ell}{\sqrt{2}}(a + a^+) \qquad p = \frac{\hbar}{i\ell\sqrt{2}}(a - a^+)$$

$$[x, p] = i\hbar \mathbb{1} \implies [a, a^+] = \mathbb{1}$$

$$\mathcal{H} = \hbar\omega\left(a^+a + \frac{1}{2}\right) \qquad \hat{n} = \hat{a}^+\hat{a}$$

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2 \ldots$$

$$H|n\rangle = E_n|n\rangle$$

1) $a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a|0\rangle = 0$

2) $a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\{|n\rangle\}$ è ortonormale

$$\langle n|n\rangle = 1 \Leftarrow \langle 0|0\rangle$$

$$\langle n+m|n\rangle = 0 \qquad m > 0$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \qquad |n+m\rangle = \frac{1}{\sqrt{(n+m)!}}(a^\dagger)^{n+m}|0\rangle$$

$$\langle n+m| = \frac{1}{\sqrt{(n+m)!}}\langle 0|(a)^{n+m}$$

$$\frac{1}{\sqrt{(n+m)!}}\frac{1}{\sqrt{n!}}\underbrace{\langle 0|a^{n+m}\,a^{\dagger n}|0\rangle} = \langle n+m|n\rangle$$

$m > 0$

$$\underbrace{\langle 0|a^n}_{\langle n|\sqrt{n!}}\; a^m\; \underbrace{a^{\dagger n}|0\rangle}_{\sqrt{n!}\,|n\rangle}$$

$$\langle n|\underbrace{a^m|n\rangle} \qquad m>0$$

Gli elementi diagonali di a e $a^\dagger$ sono sempre nulli

$$a = \xi + \frac{\partial}{\partial \xi} \qquad \text{nelle coordinate adimensionali}$$

$$\langle n|a|n\rangle = \int d\xi\, \psi_n(\xi)\left(\xi + \frac{\partial}{\partial\xi}\right)\psi_n(\xi)$$

(a) (b)

a) $\int_{-\infty}^{+\infty} d\xi \, \underbrace{\underbrace{\varphi_n^2(\xi)}_{P} \, \underbrace{\xi}_{D}}_{D} = 0$

b) $\int d\xi \, \underbrace{\varphi_n(\xi)}_{P} \underbrace{\frac{\partial}{\partial \xi} \varphi_n(\xi)}_{D} = 0$

$D \quad P \quad = 0$

$\langle n | a | n \rangle = 0 \qquad \langle n | n-1 \rangle \sqrt{n} = 0$

$\langle n | a^m | n \rangle$

$\langle n | a^2 | n \rangle = \langle n | a \, a | n \rangle = \langle n | a | n-1 \rangle \sqrt{n}$

$= \sqrt{n} \sqrt{n-1} \, \langle n | n-2 \rangle = 0$

$\int \varphi_n(\xi) \Big(\underset{(a)}{\xi} + \underset{(b)}{\frac{\partial}{\partial \xi}} \Big) \varphi_{n-1}(\xi) \, d\xi$

a) $\int d\xi \, \underset{P}{\varphi_n(\xi)} \, \underset{D}{\xi} \, \underset{D}{\varphi_{n-1}(\xi)} \neq 0$

b) $\int d\xi \, \underset{P}{\varphi_n(\xi)} \underbrace{\frac{\partial}{\partial \xi} \varphi_{n-1}(\xi)}_{P}$

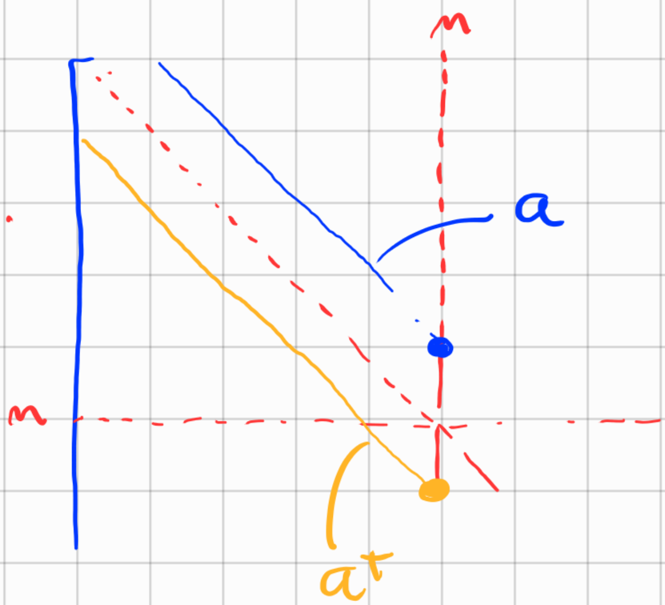
1) $a|n\rangle = \sqrt{n}\,|n-1\rangle$

R C

- $\langle n-1|a|n\rangle = \sqrt{n}$

2) $a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

- $\langle n+1|a^{\dagger}|n\rangle = \sqrt{n+1}$



$$\tilde{a} \doteq \left(\begin{array}{cccc} 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ & & & \end{array}\right.$$

$$\hat{a}^{\dagger} \doteq \left(\begin{array}{ccc} 0 & 0 & \\ \sqrt{1} & 0 & \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{3} \end{array}\right.$$

$$\hat{x} = \frac{\ell}{\sqrt{2}}(a + a^\dagger) \qquad \ell^2 = \frac{\hbar}{m\omega}$$

$$\hat{x} \doteq \frac{\ell}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ 0 & \sqrt{2} & 0 & \ddots & \\ & & \ddots & & \\ & & & & \end{pmatrix}$$

tridiagonale nella base degli stati ad H (dell'oscill. arm)

$$p = \frac{\hbar}{i\sqrt{2}\ell}(a - a^\dagger)$$

$$\hat{p} \doteq \frac{\hbar}{\sqrt{2}\ell} \begin{pmatrix} 0 & -i\sqrt{1} & & \\ i\sqrt{1} & 0 & -i\sqrt{2} & \\ & i\sqrt{2} & 0 & -i\sqrt{3} \\ & & i\sqrt{3} & 0 \end{pmatrix}$$

$$\langle 0 | a^m | 0 \rangle = 0$$

$$\langle n | a^m | n \rangle = 0$$

Meccanica Classica

eq del moto quadratiche $\begin{cases} a \\ a^+ \end{cases}$

MQ

eq Schröd indip dot !

$$\ell^2 = \frac{\hbar}{m\omega}$$

Equazioni di Hamilton in MQ

(pittura formalismo di Heisenberg)

$$A_H(t) = \underbrace{e^{\frac{i}{\hbar}Ht}}_{S^+} A \underbrace{e^{-\frac{i}{\hbar}Ht}}_{S}$$

H H

$$\begin{cases} \dot{x} = \dfrac{p}{m} \\ \dot{p} = -\dfrac{\partial V}{\partial x} \end{cases} \qquad H = \frac{p^2}{2m} + V(x)$$

Oscillatore Armonico (Heisenberg)

$$\begin{cases} \dot{\hat{x}} = \dfrac{\hat{p}}{m} \\ \dot{\hat{p}} = -m\omega^2 \hat{x} \end{cases} \qquad \begin{cases} \dot{a} = \dfrac{i}{\hbar}[H, a] \\ \dot{a}^+ = \dfrac{i}{\hbar}[H, a^+] \end{cases}$$

nel formalismo di Heisenberg

$$\hat{x}(t) = \frac{\ell}{\sqrt{2}}\left(a(t) + a^+(t)\right)$$

Heisenberg

$$\hat{p}(t) = \frac{\hbar}{i\ell\sqrt{2}}\left(a(t) - a^+(t)\right)$$

$$x = \frac{\ell}{\sqrt{2}}(a + a^+)$$

Schrödinger

$$p = \frac{\hbar}{i\ell\sqrt{2}}(a - a^+)$$

$$|\psi, t\rangle = e^{-\frac{i}{\hbar}Ht}|\psi, 0\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\psi, t\rangle = H|\psi, t\rangle$$

$$\underbrace{\langle \varphi, t | A | \varphi, t \rangle}_{\text{Scröd}} = \underbrace{\langle \varphi, 0 | e^{\frac{i}{\hbar}Ht} A e^{-\frac{i}{\hbar}Ht} | \varphi \rangle}_{\text{Heisenberg}}$$

$$A_H = e^{\frac{i}{\hbar}Ht} A e^{-\frac{i}{\hbar}Ht}$$

$$\frac{dA_H}{dt} = \frac{d}{dt}\left(e^{\frac{i}{\hbar}Ht} A e^{-\frac{i}{\hbar}Ht}\right) =$$

$$\overbrace{\left(\frac{d}{dt} e^{\frac{i}{\hbar}Ht}\right) A e^{-\frac{i}{\hbar}Ht}} +$$

$$+ \underbrace{e^{\frac{i}{\hbar}Ht} A \left(\frac{d}{dt} e^{-\frac{i}{\hbar}Ht}\right)}$$

$$= \frac{i}{\hbar} H A_H - \frac{i}{\hbar} A_H H = \frac{i}{\hbar}[H, A]$$

$$\frac{d}{d\alpha} e^{\alpha \hat{A}} = \hat{A} e^{\alpha \hat{A}} = e^{\alpha \hat{A}} \hat{A}$$

$$[H,a] = [\hbar\omega(a^\dagger a + \tfrac{1}{2}), a] =$$

$$= \hbar\omega\,[a^\dagger a, a]$$

$$[a^\dagger a, a] = a^\dagger \cancel{[a,a]} + \underbrace{[a^\dagger, a]}_{-1} a$$

$$\boxed{[a, a^\dagger] = 1}$$

$$\begin{cases} \dfrac{da}{dt} = \dfrac{i}{\hbar}(\hbar\omega)(-a) = -i\omega a \\[2ex] \dfrac{da^\dagger}{dt} = i\omega a^\dagger \end{cases}$$

$$\boxed{\begin{cases} a(t) = e^{-i\omega t} a(0) \\ a^\dagger(t) = e^{i\omega t} a^\dagger(0) \end{cases}}$$

$$\boxed{\begin{aligned} \hat{X}(t) &= \frac{\ell}{\sqrt{2}}\left(e^{-i\omega t}\hat{a}(0) + e^{i\omega t}\hat{a}^\dagger(0)\right) \\ \hat{P}(t) &= \frac{\hbar}{i\sqrt{2}\ell}\left(e^{-i\omega t}\hat{a}(0) - e^{i\omega t}\hat{a}^\dagger(0)\right) \end{aligned}}$$

a)

$$\boxed{\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\ell} + i\frac{\ell\hat{p}}{\hbar}\right) \quad b)$$
$$\hat{a}^{+} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{\ell} - i\frac{\ell\hat{p}}{\hbar}\right)}$$

$$\boxed{\begin{aligned} \hat{X}(t) &= \hat{X}(0)\cos(\omega t) + \frac{\hat{p}(0)}{m\omega}\sin(\omega t) \quad c) \\ \hat{p}(t) &= \hat{p}(0)\cos(\omega t) - m\omega\hat{X}(0)\sin(\omega t) \end{aligned}}$$

formalmente indip ℓ

Esercizio

determinare l'evoluzione temporale degli elementi di matrice

dati

$\langle 0|X|0\rangle \qquad \langle 0|X(t)|0\rangle$

$\boxed{\langle 0|X|1\rangle} \qquad \boxed{\langle 0|X(t)|1\rangle}$

$\longrightarrow$

$\boxed{\langle 1|X|0\rangle} \qquad \boxed{\langle 1|X(t)|0\rangle}$

$\langle 1|X|1\rangle \qquad \langle 1|X(t)|1\rangle$

$$\langle 0|X(t)|0\rangle = \frac{e}{\sqrt{2}}\langle 0|\left(a(t)+a^\dagger(t)\right)|0\rangle$$

$$= \frac{e}{\sqrt{2}}\left(e^{-i\omega t}\langle 0|\cancel{a}|0\rangle + e^{i\omega t}\underbrace{\langle 0|a^\dagger|0\rangle}_{\langle 0|1\rangle}\right) = 0$$

$$\hat{a}(t) = e^{-i\omega t}\hat{a}(0) \qquad \hat{a}^\dagger(t) = e^{i\omega t}\hat{a}^\dagger(0)$$

$$\langle 1|X(t)|1\rangle = \frac{e}{\sqrt{2}}\left(e^{-i\omega t}\langle 1|\underbrace{a|1\rangle}_{|0\rangle} + e^{+i\omega t}\langle 1|\underbrace{a^\dagger|1\rangle}_{\sqrt{2}|2\rangle}\right) = 0$$

$\langle 1|0\rangle$ $\qquad$ $\langle 1|2\rangle$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\langle 0|\hat{X}(t)|1\rangle = \frac{e}{\sqrt{2}}\left(e^{-i\omega t}\underbrace{\langle 0|\underbrace{a|1\rangle}_{|0\rangle}}_{1} + \cancel{e^{i\omega t}\langle 0|\underbrace{a^\dagger|1\rangle}_{\sqrt{2}|2\rangle}}\right)$$

$$\boxed{\langle 0|\hat{X}(t)|1\rangle = \frac{e}{\sqrt{2}}e^{-i\omega t}}$$

$$\boxed{\langle 1|\hat{X}(t)|0\rangle = \frac{e}{\sqrt{2}}e^{i\omega t}}$$

Esercizio

Conoscendo i valori medi su di uno stato generico $|\psi\rangle$

$$\langle x(0)\rangle = \langle\psi|\hat{x}|\psi\rangle$$

$$\langle p(0)\rangle = \langle\psi|\hat{p}|\psi\rangle$$

determinare $\langle x(t)\rangle$ e $\langle \hat{p}(t)\rangle$

teorema di Ehrenfest

$$\begin{cases} \dot{x} = \dfrac{p}{m} \\ \dot{p} = -\dfrac{\partial V}{\partial x} \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\langle x\rangle = \dfrac{1}{m}\langle p\rangle \\ \dfrac{d}{dt}\langle p\rangle = -\left\langle \dfrac{\partial V}{\partial x}\right\rangle \end{cases}$$

$$\left\langle \frac{\partial V}{\partial x}\right\rangle = \frac{\partial V(\langle x\rangle)}{\partial \langle x\rangle}$$

se $\boxed{V(x) = a + bx + cx^2 + dx^3}$

$$\frac{\partial V}{\partial x} = b + 2cx + 3dx^2 \qquad \left\langle \frac{\partial V}{\partial x}\right\rangle = b + 2c\langle x\rangle + 3d\langle x^2\rangle$$

$$V(\langle x\rangle) = a + b\langle x\rangle + c\langle x\rangle^2 + d\langle x\rangle^3$$

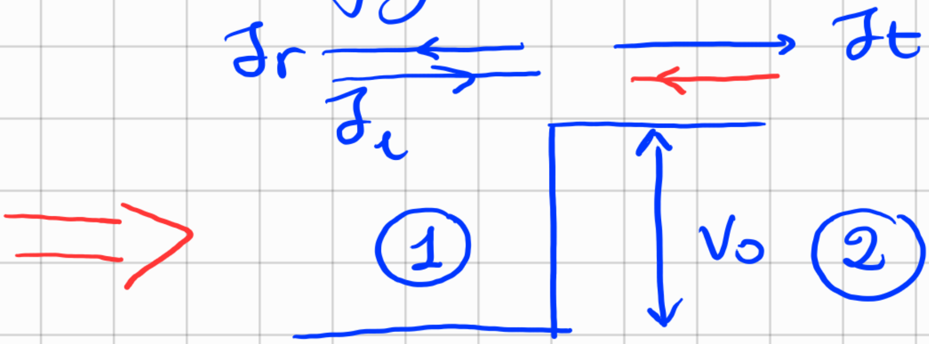
$$\frac{\partial V(\langle x\rangle)}{\partial\langle x\rangle} = b + c\langle x\rangle + 3d\langle x\rangle^2 = \left\langle \frac{\partial V}{\partial x} \right\rangle$$

$$\frac{d}{dt}\langle x(t)\rangle = \frac{1}{m}\langle p(t)\rangle$$

$$\frac{d}{dt}\langle p(t)\rangle = -m\omega^2\langle x(t)\rangle + 3d\langle x^2(t)\rangle$$

$$\boxed{\begin{aligned} \langle x(t)\rangle &= \langle x(0)\rangle \cos\omega t + \frac{\langle p(0)\rangle}{m\omega}\sin\omega t \\ \langle p(t)\rangle &= \langle p(0)\rangle \cos\omega t - m\omega\langle x(0)\rangle \sin\omega t \end{aligned}}$$

Diffusione



$E > V_0$

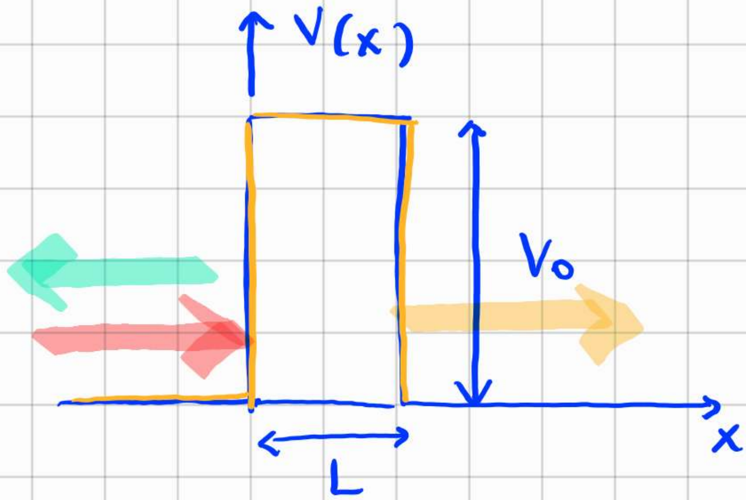
$$\psi_I = A e^{ikx} + B e^{-ikx}$$

$$\psi_{II} = A' e^{iqx} + \cancel{B' e^{-iqx}}$$

$$k = \sqrt{\frac{2m}{\hbar^2} E}$$

$$q = \sqrt{\frac{2m}{\hbar^2} (E - V_0)}$$

Diffusione da una barriera di potenziale



1) Effetto tunnel

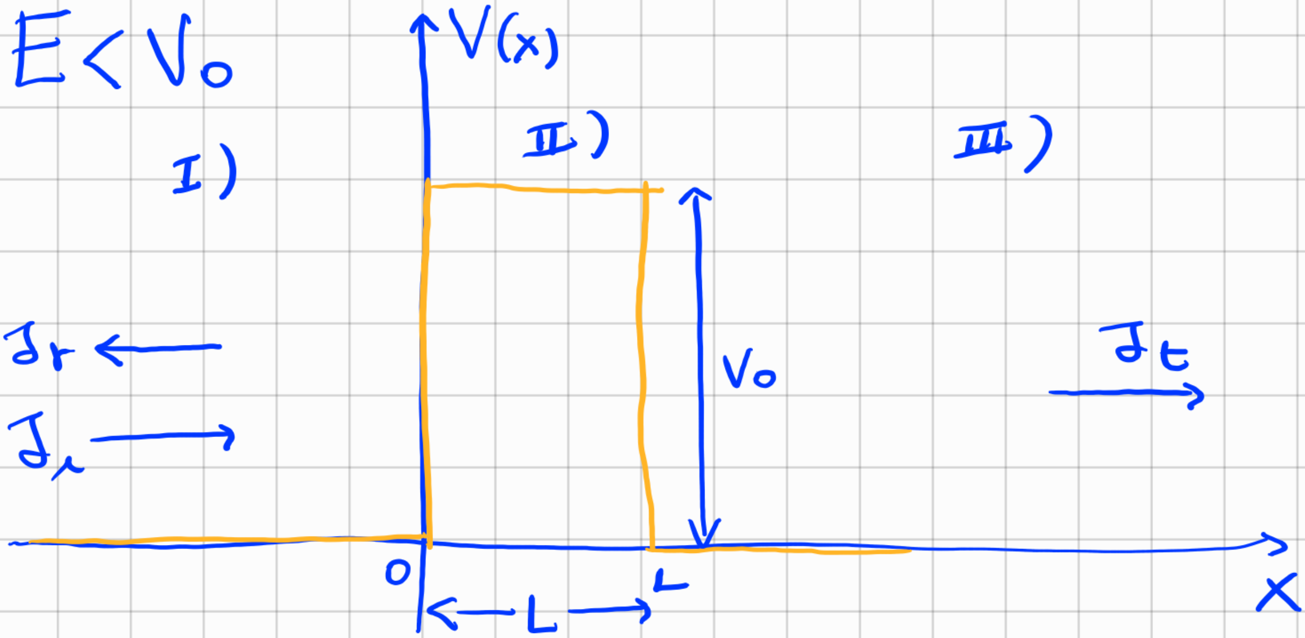
$E < V_0$ una certa corrente è TRASMESSA

2) Diffusione risonante

$E > V_0$ la lunghezza d'onda di De Broglie delle particelle incidenti se accordata o meno con la lunghezza della barriera L

T_{max}

$E < V_0$



$$T = \frac{J_t}{J_i} \qquad R = \frac{J_r}{J_i} = 1 - T$$

I) onde $k = \sqrt{\frac{2m}{\hbar^2} E}$ $\quad e^{\pm ikx}$

III) onde $k = \sqrt{\frac{2m}{\hbar} E}$ $\quad e^{\pm ikx}$

II) $E < V_0$ onde evanescenti $e^{\pm \gamma x}$

$$+\frac{\hbar^2}{2m} \psi_{II}'' + V_0 \psi_{II} = E \psi_{II}$$

$$\gamma = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$$

$$\psi_{II}'' = \frac{2m}{\hbar^2}(V_0 - E)\, \psi_{II}$$

è positivo

I) $\psi_I = A e^{ikx} + B e^{-ikx}$

II) $\psi_{II} = C e^{-\gamma x} + D e^{\gamma x}$

III) $\psi_{III} = E e^{ikx} + \cancel{F e^{-ikx}}$

la ψ_{II} è normalizzabile nella regione II) il coeff D è $\neq 0$!

$T = \frac{J_t}{J_i}$ $J_i = \frac{\hbar k}{m} |A|^2$ $J_t = \frac{\hbar k}{m} |E|^2$

$T = \frac{|E|^2}{|A|^2}$? $\Leftarrow$ condizioni di continuità in $x=0$ e $x=L$

Condizioni di Raccordo

$$x=0 \quad \begin{cases} \psi_I(0)=\psi_{II}(0) \\ \psi'_I(0)=\psi'_{II}(0) \end{cases}$$

$$A+B = C+D = Z_1 E$$

$$ik(A-B) = -\gamma(C-D)$$

$$A-B = i\frac{\gamma}{k}(C-D) = i\frac{\gamma}{k} Z_2 E$$

$$x=L$$

$$\overbrace{Ce^{-\gamma L}}^{C'} + \overbrace{De^{\gamma L}}^{D'} = \overbrace{Ee^{ikL}}^{E'}$$

$$-\gamma\left(Ce^{-\gamma L} - De^{\gamma L}\right) = ikEe^{ikL}$$

$$C'+D'=E'$$

$$C'-D' = -\frac{ik}{\gamma}E'$$

$$C' = \frac{1}{2}\left(1 - i\frac{k}{\gamma}\right)E'$$

$$D' = \frac{1}{2}\left(1 + i\frac{k}{\gamma}\right)E'$$

$$C = \frac{1}{2}\left(1 - i\frac{k}{\gamma}\right)e^{(ik+\gamma)L}E$$

$$D = \frac{1}{2}\left(1 + i\frac{k}{\gamma}\right)e^{(ik-\gamma)L}E$$

$$C+D=\underbrace{\left\{\frac{1}{2}\left(1-i\cancel{\frac{\gamma}{\gamma}}\right)e^{(ik+\gamma)L}+\frac{1}{2}\left(1+i\cancel{\frac{\gamma}{\gamma}}\right)e^{(ik-\gamma)L}\right\}}_{Z_1}E$$

$$C-D=\underbrace{\left\{\frac{1}{2}\left(1-i\cancel{\frac{\gamma}{\gamma}}\right)e^{(ik+\gamma)L}-\frac{1}{2}\left(1+i\cancel{\frac{\gamma}{\gamma}}\right)e^{(ik-\gamma)L}\right\}}_{Z_2}E$$

$E \ll V_0$

$$C+D=Z_1E \qquad C-D=Z_2E$$

$$A+B=Z_1E \qquad +$$

$$A-B=i\frac{\gamma}{k}Z_2E \qquad =$$

$$A=\frac{1}{2}\left(Z_1+i\frac{\gamma}{k}Z_2\right)E$$

$$\boxed{T_t=\frac{|E|^2}{|A|^2}=\frac{1}{\frac{1}{4}\left|Z_1+i\frac{\gamma}{k}Z_2\right|^2}}$$

$$q^2 = \frac{2m}{\hbar^2} V_0 \qquad k^2 = \frac{2m}{\hbar^2} E \qquad \gamma^2 = \frac{2m}{\hbar^2}(V_0 - E)$$

$$\boxed{T = \frac{1}{1 + \dfrac{(q/k)^2}{4\left[1 - (k/q)^2\right]} \, sh^2(\gamma L)}}$$

$$sh^2(\gamma L) = \left(\frac{e^{\gamma L} - e^{-\gamma L}}{2}\right)^2 \xrightarrow[\gamma L \gg 1]{} \frac{e^{2\gamma L}}{4}$$

$E \ll V_0$ a L fissato

$$\gamma/k \gg 1 \qquad \frac{V_0}{E} \gg 1$$

$$Z_1 \simeq \frac{1}{2} e^{(ik+\gamma)L} + \frac{1}{2} e^{(ik-\gamma)L} = e^{ikL} ch(\gamma L)$$

$$Z_2 = \frac{1}{2} e^{(ik+\gamma)L} - \frac{1}{2} e^{(ik-\gamma)L} = \underbrace{e^{ikL}}_{\text{fase}} sh(\gamma L)$$

$$\boxed{T = \frac{1}{\frac{1}{4}\left|\cancel{Z_1} + i\frac{\gamma}{k} Z_2\right|^2} \simeq \frac{1}{\frac{1}{4}\frac{\gamma^2}{k^2} sh^2(\gamma L)}}$$

$$\frac{\gamma}{k} \gg 1$$

$$\gamma^2 = \frac{2m}{\hbar^2}(V_0 - E) \qquad \gamma L \gg 1$$

$$T \simeq \frac{1}{\frac{\gamma^2}{4k^2}\,\frac{e^{2\gamma L}}{4}} \simeq \frac{16E}{V_0}\, e^{-2\gamma L}$$

$$2\gamma L = 2\sqrt{\frac{2m}{\hbar^2}(V_0 - E)\,L^2} =$$

$$= 2\,\frac{\sqrt{2m(V_0-E)L^2}}{\hbar}$$

allora $\simeq \sqrt{2mV_0L^2}$

$\hbar$

$\hbar \ll \sqrt{2mV_0L^2} \quad T \sim 0$

T dip expon $\sqrt{V_0}$

dipe expon L

J_t dipe expon da V_0