Boltzmann's Equation as a Master Equation

The Master Equation

Consider the occupation number of a discrete set of states $f(n,t) = \#$ of systems in state n

The equation of motion for $f(n,t)$ is called Master Equation and it is a balance equation

$$\frac{d f(n,t)}{dt} = N_{in} - N_{out}$$

increase rate
\# of systems that $\to n$ in unit time

decrease rate
\# of systems that $n \to$ (escape from n) in the unit time

$$N_{in} = \sum_k f(k,t)\, W_{k\to n}$$

$k \to n$

transition rates

$$N_{out} = \sum_k f(n,t)\, W_{n\to k}$$

$n \to k$

$$[W] = \frac{1}{T}$$

$$\frac{d f(n,t)}{dt} = \sum_k f(k,t)\, W_{k\to n} - f(n,t) \sum_k W_{n\to k}$$

suppose that $W_{k\to n} = 0$ then the population of level n will decrease exponentially

Now consider instead of discrete state n a continuous variable x then $f(x,t)$ can be a probability density (or a density ...)

$$\frac{\partial f(x,t)}{\partial t} = \underbrace{\int dx' f(x',t) \underbrace{W(x|x')}_{\substack{\text{transition} \\ \text{rates from} \\ x' \to x}}}_{N_{in}(x)} - \underbrace{\int dx' f(x,t) \underbrace{W(x'|x)}_{\substack{\text{transition} \\ \text{rate from} \\ x \to x'}}}_{N_{out}(x)}$$

now $[W] = \frac{1}{T[x]}$

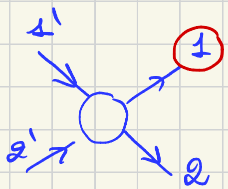
The Boltzmann's Equation

let $f(\vec{p}_1, t)$ the distribution of momenta $(\vec{p}_i)$ at time t

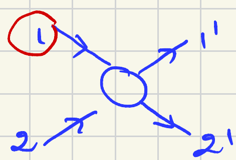
$$\frac{\partial f(\vec{p}_1, t)}{\partial t} = N_{in}(\vec{p}_1, t) - N_{out}(\vec{p}_1, t)$$

In and Out processes arise from the scattering between 2 particles (low density limit of BBGKY)

IN



OUT



$1 \equiv \vec{p}_1$
$2 \equiv \vec{p}_2$
$1' \equiv \vec{p}_1{}'$
$2' \equiv \vec{p}_2{}'$

here to get N_{in} we should integrate on momenta $1', 2', 2$ also the scattering rates depends on four momenta $1\,2\,1'\,2'$

$$N_{in}(1) = \int d1' \int d2\,d2'\, f^{(2)}(1'2')\, W_{1'2' \to 12}$$

distribution of 2 momenta 1'2'

$$N_{out}(1) = \int d1' \int d2\,d2'\, f^{(2)}(12)\, W_{12 \to 1'2'}$$

In the stosszahlansatz approximation

$$f^{(2)}(12,t) = f^{(1)}(1,t)\, f^{(1)}(2,t)$$

$$f^{(2)}(1'2',t) = f^{(1)}(1't)\, f^{(1)}(2't)$$

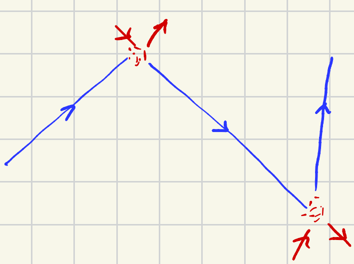
Molecular chaos approximation

valid when interparticle distance is large compared with interaction range

$$\frac{a}{e} \ll 1$$

a ← interaction range

e ← interparticle distance



each shock can be regarded as independent event

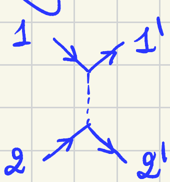
$$N_{in}(1,t) = \int d1' f(1',t) \underbrace{\int d2\, d2' f(2',t)\, W_{1'2' \to 12}}_{W(1|1')}$$

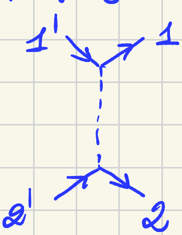
$$N_{out}(1,t) = f(1,t) \int d1' \underbrace{\int d2\, d2' f(2,t)\, W_{12 \to 1'2'}}_{W(1'|1)}$$

The Boltzmann's Eq is thus

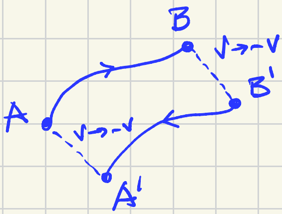
$$\frac{\partial f(1)}{\partial t} = \int d1' d2' d2 \left[f(1') f(2') W_{1'2' \to 12} - f(1) f(2) W_{12 \to 1'2'} \right]$$

and by Parity and Time-Reversal symmetries of the scattering rates

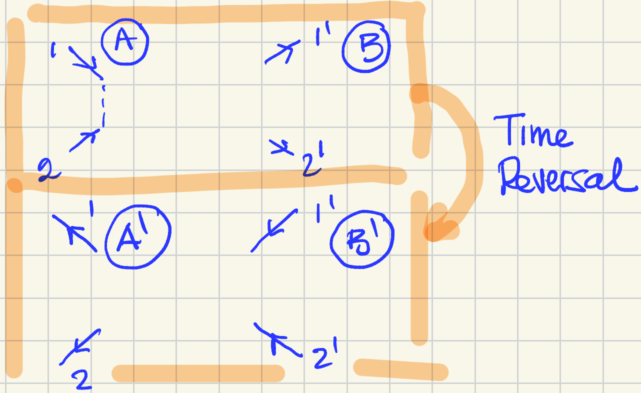




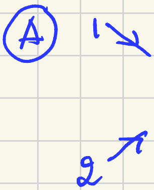
$$W_{12 \to 1'2'} = W_{1'2' \to 12}$$



A
B
B'
A'
v -> -v
v -> -v



A
B
A'
B'
1
2
1'
2'
Time Reversal



A
1
2