Eigenstate thermalization Hypothesis (ETH)

Josh Deutsch (1991) Mark Srednicki (1994)

a quantum isolated system ~~is being~~ has a time evolution which is formally known

$$|\psi, t\rangle = \sum_\alpha C_\alpha e^{-\frac{i}{\hbar}E_\alpha t} |\alpha\rangle$$

where

$H|\alpha\rangle = E_\alpha|\alpha\rangle$ are eigenstates of the N body system

Consider a quantum average of an observable

$$\langle\psi, t|\hat{A}|\psi, t\rangle = \sum_{\alpha\beta} C^*_\alpha C_\beta e^{+\frac{i}{\hbar}(E_\alpha - E_\beta)t} \langle\alpha|\hat{A}|\beta\rangle$$

if E_α, E_β are discrete and nondegenerate the behaviour is oscillatory

⇒ Problem How ISOLATED quantum systems thermalizes

⇒ Do thermalization depends on our measurements i.e. a thermal state exists only after observation and it is not a property of the quantum system in itself?

Consider the time-average

$$\bar{A} \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T}\int_0^T dt' \langle \psi,t|\hat{A}|\psi,t\rangle$$

this can be done explicitly in QM !!

$$\bar{A} = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt' \Big(\underbrace{\Sigma_\alpha |C_\alpha|^2 \langle\alpha|A|\alpha\rangle}_{\text{independent on time}} + \underbrace{\sum_{\alpha\neq\beta} C_\alpha^* C_\beta \langle\alpha|A|\beta\rangle e^{\frac{i}{\hbar}(E_\alpha - E_\beta)t}}_{\text{off-diagonal therms do depend on time}} \Big)$$

Vanishing of off diagonal therms

$$\bar{A} = \Sigma_\alpha |C_\alpha|^2 \langle\alpha|A|\alpha\rangle + \sum_{\alpha\neq\beta} \frac{C_\alpha^* C_\beta}{\frac{i}{\hbar}(E_\alpha - E_\beta)} \langle\alpha|\hat{A}|\beta\rangle \underbrace{\frac{e^{\frac{i}{\hbar}(E_\alpha-E_\beta)T} - 1}{T}}_{T\to\infty}$$

for a large system E_α and E_β could be very small e.g. for perfect gas 2 single particle level ~~as~~ can be as ~~near~~ close as $1/V^{1/3}$ where V is the volume

Single particle $E_{\vec{k}} = \frac{\hbar^2 k^2}{2m}$ $\quad K_n^\alpha = \frac{n\pi}{L_\alpha}$ $\quad \alpha = 1,3$

thus a single particle excitation may have

$$E_\beta - E_\alpha \sim \frac{\hbar^2}{m}\Delta K \sim \frac{1}{L}$$

We have to make further assumptions on the properties of A in particular we can assume that for a $N \to \infty$ particle system

$$\langle \alpha | A | \beta \rangle = A_{\alpha\alpha} \delta_{\alpha\beta} + (1-\delta_{\alpha\beta}) A_{\alpha\beta}$$

weakly dependent on E_α that is

$A_{\alpha+1,\alpha+1} - A_{\alpha,\alpha}$ is

exponentially small with the system size!

if this is true

$$\frac{\langle \alpha | A | \beta \rangle}{E_\alpha - E_\beta}$$

for $\alpha \neq \beta$ is exponentially small in the system size and we do not wait the time average time of this part that should fulfill

$$|E_\alpha - E_\beta| T \gg 1$$

for a large system it is just the <u>quantum average</u> that makes the job

<u>without making the time average!</u>

However we have

$$\bar{A} \simeq \sum_\alpha |c_\alpha|^2 \langle\alpha|\hat{A}|\alpha\rangle$$

but this will depend on initial data trough c_α!

For an isolated system however the $|c_\alpha|^2$ could not be different from zero outside a small energy shell around the Energy E

say

$$|c_\alpha|^2 = \begin{cases} \neq 0 & E \leq E_\alpha \leq E+\Delta \\ = 0 & \text{elsewhere} \end{cases}$$

if $\langle\alpha|\hat{A}|\alpha\rangle$ depends weakly on Energy

$$\bar{A} = A_{typ} \sum_\alpha |c_\alpha|^2 \simeq A_{typ}$$

but A_{typ} is the MICROCANONICAL average

$$\langle A \rangle_{Micro} = \mathrm{tr}\, \rho_{micro} A = \sum_{E \leq E_\alpha \leq E+\Delta} \langle\alpha|A|\alpha\rangle \Big/ \sum_{E \leq E_\alpha \leq E+\Delta} 1 = A_{typ}$$

therefore in the ETH

$$\langle \psi, t | \hat{A} | \psi, t \rangle \simeq \bar{A} \simeq \langle A \rangle_{Micro}$$

the system looks ergodic <u>for the quantity A</u>

- Notice that in this view is the eigenstate of the energy structure that determines the properties of large systems behaviour

- Notice that ETH depends on observable A