The Kaç ring model

Is a reversible toy model
which display irreversible (?)
tendency to equilibrium

Consider N points (balls) on a circle
each point can assume 2 state
black or white
in between each point can lie
a marker

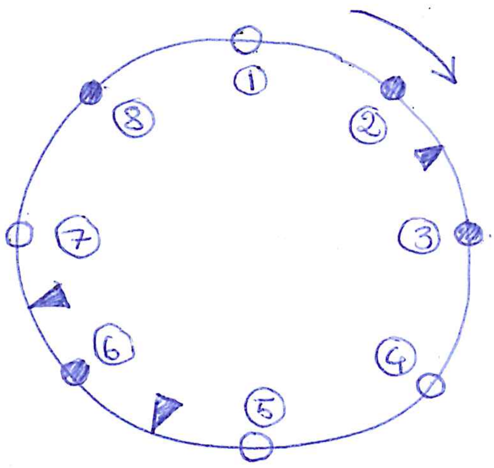


Fig 1

N points
m markers
N_b black points
N_w white points

$N = 8$
$m = 3$
$N_b = 4$
$N_w = 4$

- in one time step the circle is rotated (clockwise in the example)

  so that the point 1 assumes the position of 2

  2 → 3   3 → 4 ...

- if a point ~~has~~ crosses a marker change state

So that starting from the previous situation the first step gives

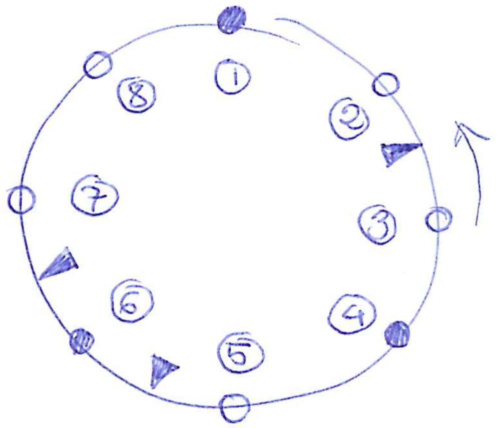
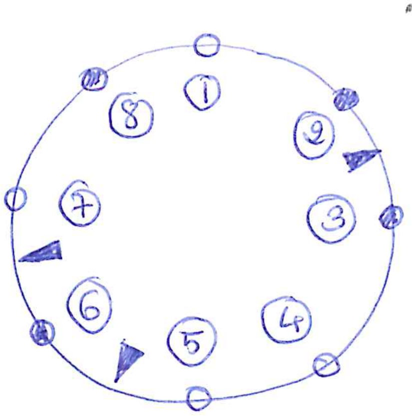


Fig 2

the model is reversible (let's step back...



which is the original configuration...

... repeat for n step...

Fig 3

let

$m_B = $ # of black points with a marker just in front

$m_W = $ # of white points

in Fig 1

$m_B = 2$

$m_W = 1$

then the evolution of white and black ~~dots~~ number of points is

$$N_W(t+1) = N_W(t) + m_B(t) - m_W(t)$$

$$N_B(t+1) = N_B(t) + m_W(t) - m_B(t)$$

We have 2 conservation laws

$N_W + N_B = N$ total # of points

$m_W + m_B = m$ total # of markers

valid at any time

let $\Delta N(t) = N_B(t) - N_W(t)$ then

$\Delta m(t) = m_B(t) - m_W(t)$

$$\boxed{\Delta N(t+1) = \Delta N(t) - 2\Delta m(t)}$$

let's consider the molecular chaos approximation (Stosszahlansatz)

$$\boxed{\text{no correlation between the color of a point and the presence of a marker just in front of it}}$$

$\Downarrow$

$$\begin{pmatrix}\text{probability that a black point has a marker just in front}\end{pmatrix} = \begin{pmatrix}\text{probability of a marker}\end{pmatrix} \times \begin{pmatrix}\text{probability of a black point}\end{pmatrix}$$

$\Downarrow$

$$\frac{m_B}{N} = \frac{m}{N} \frac{N_B}{N}$$

$$\frac{\Delta m}{N} = \frac{m}{N} \frac{\Delta N}{N}$$

μ

$$\boxed{\Delta N(t+1) = (1-2\mu)\,\Delta N(t)}$$

$\mu = \frac{m}{N}$ density of markers

$\Delta N(1) = (1-2\mu)\,\Delta N(0)$

$\Delta N(2) = (1-2\mu)\,\Delta N(1) = (1-2\mu)^2 \Delta N(0)$

$\vdots$

$$\boxed{\Delta N(t) = (1-2\mu)^t \Delta N(0)}$$

with ~~discrete~~ discrete time $t = 0, 1, 2 \ldots$

this is an exponential decay

$$\boxed{\begin{array}{l} \Delta N(t) = e^{-t/\tau} \Delta N(0) \\ \tau = 1/\ln\left(\frac{1}{1-2\mu}\right) \quad \text{if } \mu < 1/2 \end{array}}$$

if $\mu > 1/2$ $\quad 1-2\mu < 0$ and $2\mu - 1 > 0$

then we can write

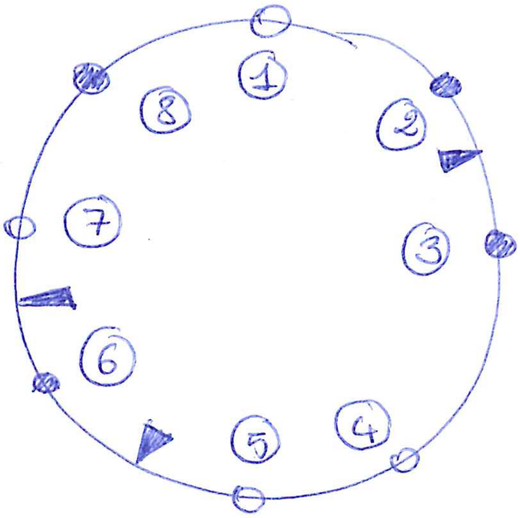
$\Delta N(t) = (-1)^t (2\mu-1)^t \Delta N(0)$

$\Delta N(t) = (-1)^t e^{-t/\tau} \Delta N(0)$ oscillatory

$\tau = 1/\ln\left(\frac{1}{2\mu-1}\right)$

This exponential beaviour is in contraddiction with the existence in Kac ring model of a recurrence time therefore both Zermelo & Loschmidt's paradoxes can be involked in this case

Numerical solution of the model



	ε	η
1	1	-1
2	-1	1
3	1	1
4	1	-1
5	-1	-1
6	-1	1
7	1	-1
8	1	1

define <u>point</u> variables $\eta_i = \begin{cases} 1 & \text{black} \\ -1 & \text{white} \end{cases}$

define <u>marker</u> variable $\varepsilon_i = \begin{cases} -1 & \text{marker present} \\ 1 & \text{marker absent} \end{cases}$

evolution equations are

$$\boxed{\eta_i(t+1) = \varepsilon_{i-1}\,\eta_{i-1}(t)}$$

$$N_B(t) = \frac{1}{2}\sum_{i=1}^{N}\left(1+\eta_i(t)\right)$$

$$N_W(t) = \frac{1}{2}\sum_{i=1}^{N}\left(1-\eta_i(t)\right)$$

$$\Delta N(t) = \sum_{i=1}^{N}\eta_i(t) = \sum_{i=1}^{N}\overbrace{\varepsilon_{i-1}\,\varepsilon_{i-2}\cdots\varepsilon_{i-t}}^{t\ \text{terms}}\,\eta_{i-t}(0)$$

$$\eta_i(t+1) = \varepsilon_{i-1}\,\eta_{i-1}(t) = \varepsilon_{i-1}\,\varepsilon_{i-2}\,\eta_{i-2}(t-1) = \varepsilon_{i-1}\,\varepsilon_{i-2}\,\varepsilon_{i-3}\,\eta_{i-3}(t-2)$$

and periodic boundary conditions must be implemented in indexes of ε

the molecular chaos hypothesis means taking the average over ε_k of the product

$$\underbrace{\langle \varepsilon_{i-1} \varepsilon_{i-2} \cdots \varepsilon_{i-t} \rangle}_{t \text{ terms uncorrelated}} = f(t) \quad \text{independent on } i$$

$$\varepsilon_1 \varepsilon_2 \cdots \varepsilon_t = (-1)^s$$

s = # of markers between t points

$$\underbrace{P(s) = \binom{t}{s} \mu^s (1-\mu)^{t-s}}_{\text{probability of } s \text{ markers in } t \text{ points}} \quad \text{binomial}$$

$$\langle \varepsilon_1 \varepsilon_2 \cdots \varepsilon_t \rangle = \sum_{s=0}^{t} (-1)^s P(s) = (1-2\mu)^t$$

c. v. d.