logistic map & chaos

We derive the logistic Map from Malthus Verhuest population dynamics model described by non-linear differential equation

$$\frac{dn}{dt} = \alpha n \left(1 - \frac{n}{k}\right)$$

$K>0$ resource exhaustion

α = reproduction rate

the non linear term $-\alpha n^2/k$ leads to deviation from exponential increase of population and describe (as $\alpha>0$ $k>0$) e.g. a resource exhaustion term

$$x = n/k$$

$$\frac{dx}{dt} = \alpha x (1-x)$$

$$\boxed{x(t) = \frac{x(0)\, e^{\alpha t}}{1 + x(0)(e^{\alpha t} - 1)}}$$

logistic curve

small times $x(0)(e^{\alpha t}-1) \ll 1$ $\quad x(t) \simeq x(0) e^{\alpha t}$ expon.

large times $x(t) \simeq \frac{x(0) e^{\alpha t}}{x(0) e^{\alpha t}} \to 1$ saturation by discretising the time (Euler...)

$$X(t+\Delta t) = X(t) + \alpha X(t)[1-X(t)]\Delta t$$

and scaling

$$X(t) = y(t)/\Lambda \qquad t+\Delta t \to t+1 \text{ discrete}$$

$$y_{t+1} = (1+\alpha\Delta t)\, y_t - \frac{\alpha\Delta t}{\Lambda} y_t^2$$

$$\begin{cases} 1+\alpha\Delta t = r \\ \dfrac{\alpha\Delta t}{\Lambda} = r \end{cases} \Rightarrow \Lambda = \frac{r-1}{r}$$

$$\boxed{y_{t+1} = r\, y_t(1-y_t)} \qquad r>0$$

(Logistic Map <u>t discrete</u>

the Logistic Map has fixed points

$$y_{t+1} = f(y_t)$$

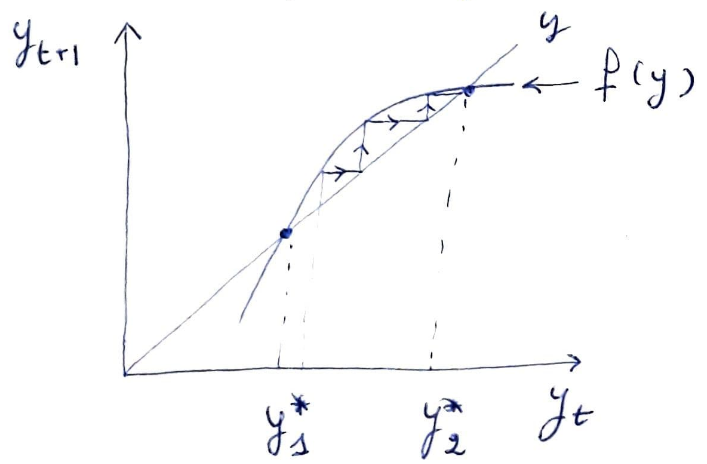
$$\boxed{y^* = f(y^*)}$$

$y^* = \frac{r-1}{r}$ nontrivial fixed point

$y^* = 0$ trivial fixed point

if a map $y_{t+1}=f(y_t)$ has a fixed point $y^*=f(y^*)$ the stability of y^* is given by

$$\left|\frac{df}{dy}\Big|_{y^*}\right|<1 \quad \text{STABLE}$$



$y_1^* \rightarrow$ UNSTABLE

$y_2^* \rightarrow$ STABLE

for logistic map

$$f(y)=ry(1-y)$$

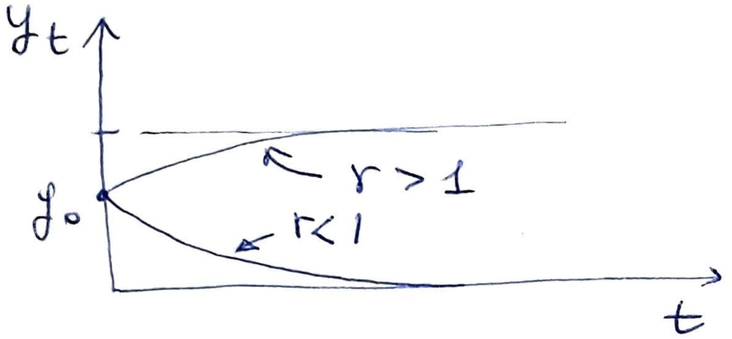
$$\frac{df}{dy}=r-2rx$$

Trivial fixed point $y^*=0$ $\qquad \frac{df}{dy}\Big|_0=r$

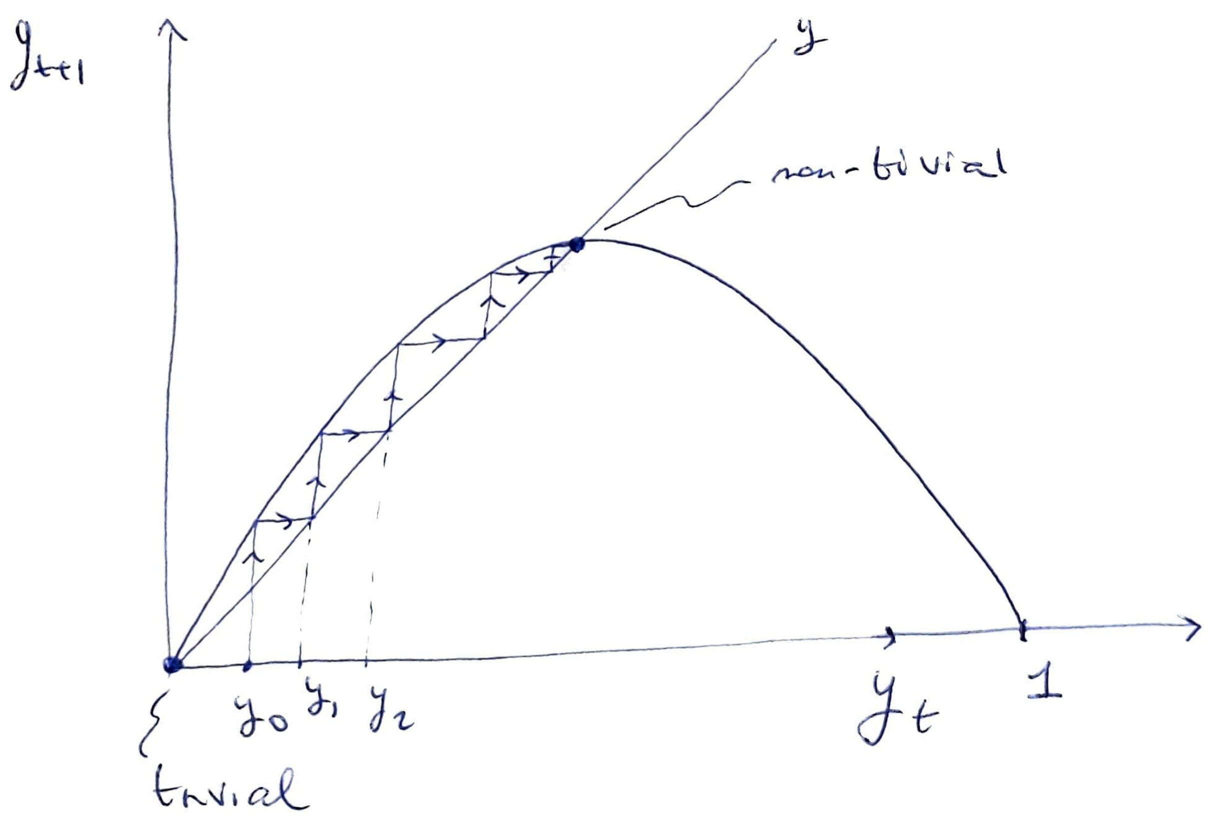
non Trivial fixed point $y^*=\frac{r-1}{r}$ $\qquad \frac{df}{dy}\Big|_{y^*}=2-r$

fixed point	$r<1$	$r>1$	$3>r>2$	$r>3$
$y^*=0$	stable	unstable	unstable	unstable
$y^*=\frac{r-1}{r}$	unstable	stable	oscilatory	

this is the prediction of local stability



Non-linear evolution.



Liapunov exponent

$$\lambda = \lim_{t \to \infty} \lim_{\delta \to 0} \frac{1}{t} \ln \frac{|x'(t) - x(t)|}{\delta}$$

$x'(0) = x(0) + \delta$ continuous time

$x'(t) \to y'_{t+1}$ ~~t~~ discrete

$x(t) \to y_{t+1}$

now let

$$y'_{t+1} = f(y_t | y'_0) = f(y_t | y_0 + \delta)$$

$$y_{t+1} = f(y_t | y_0)$$

approx

$$\frac{|x'(t) - x(t)|}{\delta} \to \left.\frac{df}{dy}\right|_{y_t}$$

average over times

$$\lambda = \lim_{t \to \infty} \frac{1}{t+1} \sum_{t'=0}^{t} \ln |f'(y_{t'})|$$

$$= \lim_{t \to \infty} \frac{1}{t+1} \sum_{t'=0}^{t} \ln |r(1 - 2x_t)|$$

$T << 1/\lambda$

write a code to evaluate iteratively

$$y_{n+1} = r\, y_n (1 - y_n)$$

and the Lyapunov exponent

$$\lambda_n = \frac{1}{n+1} \sum_{0}^{n} \ln|r(1-2y_n)|$$

as $n \to \infty$ (very large..)

perform calculation with

a) $0 < r < 1$

b) $1 < r < 2$

c) $2 < r < 3$

d) increase r for $r > 3$ smoothly

what is the result?

Discrete map is chaotic (in some region of parameter's space) while its continuous version not. However ~~some sets~~ of chaos exists in continuous time evolution