Estimate of recurrence time from Gallavotti "Trattatello di Meccanica Statistica" freely available on-line

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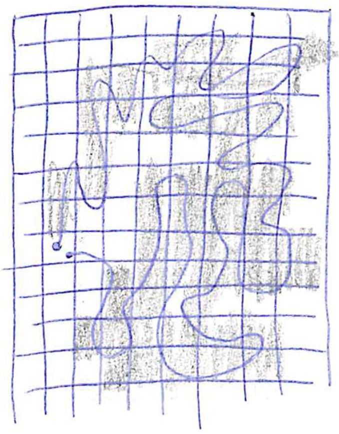
Let's consider a perfect gas at equilibrium

We can estimate the recurrence time as the time the system takes to visit all the microscopic states which statistically contributes to a macroscopic state.

Discretize the available phase space



one cell ≡ one microscopic state

# of gray cells ≡ # of microscopic states for one macroscopic state

$$T_{recurrence} = \tau Z$$

τ: time the system spend in one cell

Estimate of τ

(1)

the size of the cell in P.S. is due to uncertainty principle

$$\Delta p \Delta q \sim h \quad \text{or} \quad \Delta E \, \tau \sim h$$

for a thermalized system

$$\Delta E \sim k_B T$$

(for one particle $\Delta p \neq 0 \Rightarrow \Delta E = \frac{2 \vec{p} \cdot \Delta \vec{p}}{2m}$

but $\Delta \vec{p} = \vec{p} - \cancel{\langle \vec{p} \rangle} \Rightarrow \Delta E = \frac{|\vec{p}|^2}{m}$

on average $\langle \Delta E \rangle = \frac{\langle |p|^2 \rangle}{m} = \frac{2U}{N} = 3 k_B T$)

therefore $\boxed{\tau \sim \frac{h}{k_B T}}$

Calculation of Z

It is easy to perform this calculation in the canonical ensemble

(Gallavotti does in microcanonical)

perfect gas $Z = \frac{1}{N!} \left(\frac{V}{\lambda^3} \right)^N \quad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}$

(2)

(In canonical ensemble

$$Z = \int \frac{d^{3N}p\, d^{3N}q}{N!\, h^{3N}} e^{-\beta \sum_i \frac{p_i^2}{2m}} = \frac{V^N}{N!} \Big(\underbrace{\int_{-\infty}^{+\infty} dp\, e^{-\beta \frac{p^2}{2m}}}_{\text{a generic coordinate of } \vec{p}} \Big)^{3N})$$

Now consider the Stirling's formula for large N

$$N! = N^N e^{-N} \sqrt{2\pi N}$$

$$Z = \frac{V^N}{\lambda^N N^N} \frac{e^N}{\sqrt{2\pi N}}$$

the average interparticle distance is

$$l = \Big(\underbrace{\frac{V}{N}}_{\text{volume x particle}} \Big)^{1/3}$$

$$\boxed{Z = \frac{1}{\sqrt{2\pi N}} \left(\frac{l}{\lambda} e^{1/3} \right)^{3N}}$$

the number of microstates is exponentially large as far as

$$\frac{l}{\lambda} e^{1/3} > 1$$

$\ell/\lambda \gg 1$ is just the classical limit

$$T_{recurrence} \sim \frac{h}{K_B T} \left(\frac{1}{\sqrt{2\pi N}}\right) \left(\frac{\ell}{\lambda} e^{1/3}\right)^{3N}$$

irrelevant

when $T = 300°K$

$\tau \sim 1.6\ 10^{-13}\ s$

for H_2 $M \simeq 2 m_p$ $\lambda = 7.1 \cdot 10^{-11}\ m$

1 mole at $T = 300°K \Rightarrow 24\ lt = V$

$\ell = (V/N)^{1/3} = 3{,}4\ 10^{-9}\ m \gg \lambda$

Avogadro

$\ell/\lambda \simeq 48$ $\qquad$ $\ell/\lambda\, e^{1/3} \cong 67$

$$T_{recurrence} \simeq 1{,}6 \cdot 10^{-13} \frac{1}{2 \cdot 10^{12}} (67)^{3 \times 6 \times 10^{23}}\ s$$

compare with the universe's age $10^{17}\ s$