Introduction to dynamical systems

A **dynamical system** is a law for the evolution of the variables characterizing a system at every future times, given an initial value. It is described by a set of variables (which can be grouped into a vector to have a more compact notation)

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{pmatrix}$$

and a deterministic law describing their evolution. The set of all the $\{\mathbf{x}(t)\}$ represents all the possible states of the system and it is usually called **phase space**.

- Continuous time evolutions are described by a set of first-order differential equations

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{G}(\mathbf{x}(t)), \quad \text{given the initial conditions } \mathbf{x}(0) = \mathbf{x}_0. \tag{1}$$

  The solution is a trajectory described by $\mathbf{x}$ in the phase space that can be formally written as

$$\mathbf{x}(t) = \mathbf{S}^t \mathbf{x}(0),$$

  where $\mathbf{S}^t$ is the time-evolution operator. The function $\mathbf{G}$ represents a **velocity field** and if it is sufficiently regular, the Cauchy theorem guarantees the existece and unicity of the solution.

- Discrete time evolutions are described by a deterministic map, which is a recursion law which is the discrete version of (1)

$$\mathbf{x}(t_{n+1}) = \mathbf{G}(\mathbf{x}(t_n)) \quad \text{given the initial conditions } \mathbf{x}(0) = \mathbf{x}_0. \tag{2}$$

  The mapping is called **invertible** if the correspondence between all the possible points at time t_n and all the possible points at time $t_{n+1}$is biunivocal if it exists the inverse map $\mathbf{G}^{-1}$ such that $\mathbf{x}_{n-1} = \mathbf{G}^{-1}(\mathbf{x}(t_n))$.

Conservative and nonconservative systems

The time evolution given by (1) or (2) transforms a piece of the phase space into another. In doing so, the initial volume could change or remain the same, in this case the systemis is called *conservative.* As far as a continuous evolution is concerned, one has the following criterion:

- If $\nabla \cdot \frac{d\mathbf{x}}{dt} = \nabla \cdot \mathbf{G} = 0$ the dynamical system is conservative [1]. In this case, the velocity field behaves as an incompressible fluid. A remarkable example of conservative dynamics is the one described by Hamiltonian systems (see App. **??**).

[1] Given a compact set of initial conditionsits infinitesimal volume is

$$dV(0) = d\mathbf{x}(0),$$

- If $\nabla \cdot \mathbf{G} < 0$ the dynamical system is *dissipative.* In this case the volumes in the phase space shrink in time into a so-called **attractors**.
- If $\nabla \cdot \mathbf{G} > 0$ the motion in the phase space becomes unbounded. This is the case in which some energy is pumped into the system (for instance an explosion).

If the evolution is discrete then one must consider the Jacobian matrix $\mathbf{J}$ whose elements are $J_{ij} = \frac{\partial G_i}{\partial x_j}$.

- If $|\det \mathbf{J}| = 1$ then the map is conservative [2].
- If $|\det \mathbf{J}| < 1$ then the map is dissipative.

Even if the equations of motion are deterministic, if the initial conditions are not perfectly fixed, the evolution cannot be predicted exactly. For linear systems, solutions corresponding to very close initial conditions do not deviate very much in time, so it is possible to predict the evolution at very large times with a small error. Nonlinear system, instead, can lead to chaotic behavior, i.e. a strong sensitivity to initial conditions, so two initially very closed solutions could evolve after a short time into two completely different trajectories.

Example 0.0.1. Pendulum. Let us analize three cases

1. The equation of motion of a **simple pendulum** of length L is, in polar coordinates,

$$\ddot{\theta} = -\frac{g}{L}\sin\theta,$$

where θ is the angle between the sting and the vertical direction. The second order equation can be decomposed into

$$\begin{aligned} \dot{\theta} &= \alpha \\ \dot{\alpha} &= -\frac{g}{L}\sin\theta, \end{aligned}$$

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as the initial points evolve $\mathbf{x}(0) \to \mathbf{x}(t)$ the volume becomes

$$dV(t) = d\mathbf{x}(t) = \left|\det \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(0)}\right| dV(0) = |\det \mathbf{M}(t)|\, dV(0) = |J|\, dV(0),$$

where $\mathbf{M}(t)$ is the Jacobian matrix. The evolution of $\mathbf{M}$ is

$$\frac{d\mathbf{M}(t)}{dt} = \frac{\partial \dot{\mathbf{x}}(t)}{\partial \mathbf{x}(0)} = \frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}(0)} = \frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{x}(0)} = \frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{M}(t).$$

Let us compute the determinant of $\mathbf{M}$ using the property of the determinant

$$\frac{d}{dt}\det(\mathbf{M}) = \det \mathbf{M}\, \mathrm{tr}\left\{\mathbf{M}^{-1}\frac{d\mathbf{M}(t)}{dt}\right\},$$

so

$$\frac{dJ}{dt} = J\,\mathrm{tr}\left\{\mathbf{M}^{-1}\frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{M}(t)\right\} = J\,\mathrm{tr}\left\{\frac{\partial \mathbf{G}(\mathbf{x})}{\partial \mathbf{x}}\right\} = J\nabla\cdot\mathbf{G}(\mathbf{x}).$$

If $\nabla \cdot \mathbf{G}(\mathbf{x})$ the Jacobian does not change and the volume in phase space remains the same.

[2] The infinitesimal volume at time n is

$$dV(t_n) = dx_1(t_n)\, dx_2(t_n)\ldots,$$

after a time-step the variables evolve by the (2) so

$$dV(t_{n+1}) = dx_1(t_{n+1})\, dx_2(t_{n+1})\ldots = |\det \mathbf{J}|\, dx_1(t_n)\, dx_2(t_n)\ldots = |\det \mathbf{J}|\, dV(t_n),$$

so the volume does not change in time if $|\det \mathbf{J}| = 1$

where

$$\mathbf{G} = \begin{pmatrix} \alpha \\ -\frac{g}{L}\sin\theta \end{pmatrix}.$$

The system is conservative since

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial\theta}\alpha + \frac{\partial}{\partial\alpha}\left(-\frac{g}{L}\sin\theta\right) = 0.$$

In the phase space the orbits are closed (periodic motion).

2. The **damped pendulum** has equations

$$\begin{aligned} \dot{\theta} &= \alpha \\ \dot{\alpha} &= -\gamma\alpha - \frac{g}{L}\sin\theta, \end{aligned}$$

where $\gamma > 0$ and

$$\mathbf{G} = \begin{pmatrix} \alpha \\ -\gamma\alpha - \frac{g}{L}\sin\theta \end{pmatrix}.$$

This system is dissipative

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial\theta}\alpha + \frac{\partial}{\partial\alpha}\left(-\gamma\alpha - \frac{g}{L}\sin\theta\right) = -\gamma < 0,$$

and the attractor is a single point in the phase space $(\theta = 0, \alpha = 0)$, twhere the orbits converge.

3. The **vertically driven and damped pendulum** has equation of motion (with rescaled parameters)

$$\ddot{\theta} = -\gamma\dot{\theta} - a\sin\theta + f\cos t,$$

since the equation is not autonomous, one can decompose the equation in a system of three first-order differential equations

$$\begin{aligned} \dot{\theta} &= \alpha \\ \dot{\alpha} &= -\gamma\alpha - a\sin\theta + f\cos z \\ \dot{z} &= 1, \end{aligned}$$

$$\mathbf{G} = \begin{pmatrix} \alpha \\ -\gamma\alpha - a\sin\theta + f\cos z \\ 1 \end{pmatrix}.$$

Also in this case the system is dissipative

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial\theta}\alpha + \frac{\partial}{\partial\alpha}\left(-\gamma\alpha - a\sin\theta + f\cos z\right) + \frac{\partial}{\partial z}1 = -\gamma.$$

In this case the behavior is more complicate and depending on the parameters, it can be chaotic.

Example 0.0.2. Hénon map. This is a two-dimension map defined as

$$\begin{aligned} x(n+1) &= y(n) + 1 - ax^2(n) \\ y(n+1) &= bx(n) \end{aligned}$$

with $|b| < 1$ and

$$\mathbf{G} = \begin{pmatrix} y + 1 - ax^2 \\ bx \end{pmatrix}.$$

It is dissipative, since

$$\mathbf{J} = \begin{pmatrix} -a & 1 \\ b & 0 \end{pmatrix} \Rightarrow |\det\mathbf{J}| = |b| < 0.$$

Deterministic evolution of the pdf

The initial conditions could be also randomly distributed with a certain pdf $\rho_0(\mathbf{x})$. The deterministic evolution of $\mathbf{x}$ produces an evolution in the pdf

$$\rho_0(\mathbf{x}) \to \rho(\mathbf{x}, t).$$

Continuous evolution

Defining the **current density of probability**

$$\mathbf{J} = \rho(\mathbf{x}, t)\frac{d\mathbf{x}}{dt} = \rho(\mathbf{x}, t)\,\mathbf{G}(\mathbf{x}(t)),$$

the conservation of the probability

$$\int d\mathbf{x}\,\rho(\mathbf{x}, t) = 1,$$

implies the **continuity equation**[3][4]

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0}, \tag{3}$$

which can be written as

$$\frac{\partial \rho}{\partial t} + \mathbf{G}\cdot\nabla\rho + \rho\nabla\cdot\frac{d\mathbf{x}}{dt} = 0.$$

[3] The probability for the system to lie inside a hypervolume V is

$$P_V(t) = \int_V d\mathbf{x}\,\rho(\mathbf{x}, t),$$

after a certain time Δt

$$P_V(t + \Delta t) = P_V(t) - \int_{\Delta V} d\mathbf{x}\,\rho(\mathbf{x}, t),$$

where the last integral counts the density of points that during Δt had evolved from V outside (in this case the integral is positive) or from outside into V (in this case the integral is negative). If Δt is very small, all these points lie in a shell around the surface ∂V of V whose thickness decreases as $\Delta t \to 0$. The volume element is

$$\Delta\mathbf{x} = dS\,\Delta\ell,$$

and, introducing the versor $\hat{n}$ normal to the surface

$$\Delta\ell \simeq \frac{d\mathbf{x}}{dt}\cdot\hat{n}\Delta t.$$

For $\Delta t \to 0$

[4]

$$\Delta\mathbf{x} \to d\mathbf{x} \simeq \frac{d\mathbf{x}}{dt}\cdot\hat{n}dSdt,$$

and the integral becomes

$$\int_{\Delta V} d\mathbf{x}\,\rho(\mathbf{x}, t) \to dt\int_{\partial V}\rho(\mathbf{x}, t)\frac{d\mathbf{x}}{dt}\cdot\hat{n}dS = dt\int_{\partial V}\mathbf{J}\cdot\hat{n}dS,$$

$$\frac{d}{dt}P_V(t) = \int_V d\mathbf{x}\,\frac{\partial}{\partial t}\rho(\mathbf{x}, t) = -\int_{\partial V}\mathbf{J}\cdot\hat{n}dS = -\int_V d\mathbf{x}\,\nabla\cdot\mathbf{J}.$$

Since this relation is true for every V it can be expressed in a local form as the continuity equation

$$\int_V d\mathbf{x}\left(\frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla\cdot\mathbf{J}\right) = 0 \Longrightarrow \frac{\partial}{\partial t}\rho(\mathbf{x}, t) + \nabla\cdot\mathbf{J} = 0.$$

Conservative systems and Liouville equation

For conservative systems the continuity equation becomes

$$\frac{\partial \rho}{\partial t} + \mathbf{G} \cdot \nabla \rho = 0,$$

which describes the evolution of a density under the action of a velocity field $\mathbf{G}$ which conserves the volume of the phase space: taking an initial interval of the phase space, each point in it evolves defining a new interval which has the same volume of the initial one.

This is the case, for instance, of *Hamiltonian systems* (see App. **??**). Take, as an example, the case of a single particle in 1D, described by the two coordinates, position and momentum, in the phase space

$$\mathbf{x} = \begin{pmatrix} q \\ p \end{pmatrix},$$

with Hamiltonian

$$H = \frac{p^2}{2m} + U(q).$$

The equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m},$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{dU}{dq},$$

or, using the compact notation (1)

$$\mathbf{G} = \begin{pmatrix} \frac{p}{m} \\ -\frac{dU}{dq} \end{pmatrix}.$$

The continuity equation becomes

$$\nabla \rho \cdot \mathbf{G} = \frac{\partial \rho}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial \rho}{\partial p}\frac{\partial H}{\partial p} = \{\rho, H\},$$

which is the **Liouville equation** for the evolution of the probability density function

$$\boxed{\frac{\partial \rho}{\partial t} + \{\rho, H\} = \frac{\partial \rho}{\partial t} + \mathscr{L}\rho = 0}, \tag{4}$$

where $\mathscr{L}$ is the *Liouville operator*. This result is general for any Hamiltonian system in any dimension and number of particles. The *stationary density* is the solution of the Liouville equation with $\frac{\partial \rho}{\partial t} = 0$, so

$$\mathscr{L}\rho_{st}(\mathbf{x}) = 0,$$

which means that ρ_{st} is the eigenfunction of $\mathscr{L}$ with eigenvalue 0 and it depends explicitly on H. Notice that the Liouville equation is equivalent to say that the total derivative of ρ is zero

$$\frac{d\rho}{dt} = 0.$$

This result, known as the **Liuoville theorem** means that in a conservative system, the distribution of representative points evolves as an incompressible fluid (the volumes in the phase space are conserved). This result implies also another important theorem, the **Poincaré theorem**

Theorem 0.0.3. *Given a conservative system with a bounded phase space Γ, and a set $A \subset \Gamma$, all the trajectories starting from $x \in A$ will return back to A after some time repeatedly and infinitely many times, except for some of them in a set of zero measure.*

Discrete maps

For discrete times the evolution of the pdf is given by the **Perron-Frobenius** equation

$$\boxed{\rho_{n+1}(\mathbf{x}) = \int d\mathbf{y}\, \delta(\mathbf{x} - \mathbf{G}(\mathbf{y}))\, \rho_n(\mathbf{y}) = \mathscr{L}_{PF}\rho_n(\mathbf{x})}, \tag{5}$$

where $\mathscr{L}_{PF}$ is the *Perron-Frobenius operator*. Here the condition to impose to find the stationary distribution is

$$\mathscr{L}_{PF}\rho_{st}(\mathbf{x}) = \rho_{st}(\mathbf{x}).$$

Example 0.0.4. Take the deterministic map on $[0, 1]$

$$x_{n+1} = 4x_n(1 - x_n),$$

it is possible to verify that the invariant pdf is

$$\rho_{st}(x) = \frac{1}{\pi\sqrt{x(1-x)}}.$$

The Perron-Frobenius applied to $\rho_{st}(x)$ is

$$I(x) = \mathscr{L}_{PF}\rho_{st}(x) = \int_0^1 dy\, \frac{1}{\pi\sqrt{y(1-y)}}\delta(x - 4y(1-y)).$$

The integral can be solved introducing the function which is the argument of the delta

$$g(y) = x - 4y(1-y),$$

whose zeroes are

$$y_{1,2} = \frac{2 \pm \sqrt{4-4x}}{4} = \frac{1 \pm \sqrt{1-x}}{2},$$

and its derivative

$$g'(y) = 4(2y - 1),$$

calculated in the zeros is

$$|g'(y_1)| = |g'(y_1)| = 4\sqrt{1-x}.$$

Using the properties of the delta function, the integral gives

$$\begin{aligned} I(x) &= \frac{1}{4\pi\sqrt{1-x}}\left(\frac{1}{\sqrt{y_1(1-y_1)}} + \frac{1}{\sqrt{y_2(1-y_2)}}\right) = \\ &= \frac{1}{2\pi\sqrt{1-x}}\left(\frac{1}{\sqrt{\left(1+\sqrt{1-x}\right)\left(1-\sqrt{1-x}\right)}} + \frac{1}{\sqrt{\left(1-\sqrt{1-x}\right)\left(1+\sqrt{1-x}\right)}}\right) = \\ &= \frac{1}{\pi\sqrt{x(1-x)}} = \rho_{st}(x), \end{aligned}$$

so to say that

$$\mathscr{L}_{PF}\rho_{st}(x) = \rho_{st}(x),$$

which is the condition of stationarity.

Ensembles

Having defined a pdf $\rho(\mathbf{x},t)$ of the representative points in phase space, evolving according, for instance, to (4) or (5), it can be used to compute the averages of all the quantities one can be interested in

$$\langle A(\mathbf{x},t)\rangle = \int d\mathbf{x}\,\rho(\mathbf{x},t)\,A(\mathbf{x})\,. \tag{6}$$

This operation can be interpreted in the following way: imagine to prepare at initial time a huge number of identical systems with initial conditions chosen randomly and distributed as $\rho(\mathbf{x},0)=\rho_0(\mathbf{x})$, this defines an **ensemble**. Let the ensemble to evolve for a time t, then measure, for each element of the ensemble, the value of A. For a sufficiently large ensemble, the frequency of each value of $\mathbf{x}$ obtained will tend to $\rho(\mathbf{x},t)$ and the arithmetic average of the measures will give (6). For this reason the average computed in (6) is also called *ensemble average*.

In this approach one can ask if there is any equilibrium value ρ_{st} of the pdf and, if so, if the evolution of any ρ converges to ρ_{st}. This condition is called **mixing**

$$\rho(\mathbf{x},t)\to\rho_{st}(\mathbf{x})\,, \tag{7}$$

and is the condition at the basis of the equilibrium statistical mechanics. Actually for all practical purposes, this definition of mixing can be weaken since it is sufficient that for (almost) all observables $A(\mathbf{x})$

$$\int d\mathbf{x}\,\rho(\mathbf{x},t)\,A(\mathbf{x})\to\int d\mathbf{x}\,\rho_{st}(\mathbf{x})\,A(\mathbf{x})\,. \tag{8}$$

Another definition of mixing is the following: let us consider two sets of the phase space $\eta_1,\eta_2\in\Omega$, and consider the evolution of one of them for instance $\eta_1(t)$, remembering that the probability of a set is given by the *measure*

$$P(\eta)=\int_\eta \rho(\mathbf{x})\,d\mathbf{x},$$

then the system is mixing if

$$P(\eta_1(t)\cap\eta_2)\overset{t\to\infty}{\longrightarrow}P(\eta_1(t))\,P(\eta_2)$$

A mixing system loses the memory of the initial conditions very fast and this is s typical feature of so-called chaotic systems.

Ergodicity

The ensemble description of a system is a very natural way to study mathematically the statistical properties of the system, nevertheless its experimental "implementation" is completely infeasible (it would be quite absurd to prepare, for instance, millions of samples of the same gas to measure its pressure or its temperature). In practice, we can observe a single system during a time which is typically much larger than the characteristic times of its dynamics, performing a temporal average of its observable

$$\bar{A}(\mathbf{x})=\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}dt'\,A(\mathbf{x}(t'))\,. \tag{9}$$

It is possible to prove that limit (9) exists for almost all the possible initial conditions and under general hypothesis (**Birkhoff theorem I**).

This operation, which naturally describes a measurement process, is not always easy to compute. Luckily, in many systems, the ensemble and temporal averages are equivalent and, in this case the system is called **ergodic**

$$\mathrm{E}\left[A\left(\mathbf{x}\right)\right]=\lim_{T\to\infty}\frac{1}{T}\int_{t}^{t+T}dt'\,A\left(\mathbf{x}\left(t'\right)\right)=\int_{\Omega}d\mathbf{x}\,\rho\left(\mathbf{x}\right)A\left(\mathbf{x}\right).$$

Of course, ergodicity is possible only if the temporal average does not depend on the initial condition. Starting from an initial point an ergodic systems evolves passing on (or arbitrarily close to) all the points of the accessible phase space (which is, for a conservative system, an hypersurface with constant energy). So, a general condition (necessary and sufficient) for a system to be ergodic is that the accessible phase space (i.e. the phase space of the possible initial conditions) cannot be decomposed into two or more parts with finite measure, invariant under the time evolution (**Birkhoff theorem II**).

- An integrable system cannot be ergodic over all the phase space since its motion is constrained to lie on the invariant tori. Nonetheless, the motion can be ergodic within the single invariant tori if it is quasi-periodic.
- *It is possible to show that in general mixing implies ergodicity but ergodicity does not imply mixing.*

Generally, is not possible to determine if a system is ergodic or not. In many cases is reasonable to assume ergodicity, for example in systems composed by a huge amount of particles, and so with a lot of degrees of freedom. The *ergodic hypothesis* lies at the base of the equilibrium statistical mechanics even if in general cannot be rigorously proved.

Robustness of integrable systems

A tytpical example of non-ergodic behavior is the one of an integrable system. Remember that an Hamiltonian integrable system of N degrees of freedom (so described by $2N$ coordinates in the phace space) possesses N integrals of motion $\{I_N\}$ (where $I_1=E$ is the energy, which is always conserved even if the system is not integrable). Therefore, the Hamiltonian can be expressed as a function of the intgrals of motion **only**

$$H_0\left(I_1,\ldots,I_N\right).$$

In this case, if the motion is confined in a bounded set of the phace space, then the total motion is composed by the N independent periodic motion of the angle variables $\{\theta_j\}$ which are conjugate to the integrals of motion (action variables)

$$\theta_j\left(t\right)=\theta_0+\omega_j t.$$

The obtained trajectory lies on a N-dimensional torus and this means that the trajectory does not explore all the available phace space, and the motion is not ergodic[5]

A possible question to ask is wheter, starting from an integrable system, adding a small perturbation it is possible for the system to remain integrable. Let us consider the new Hamiltonian obtained from the original one, adding a perturbation term which depends on both the action and angle variables

$$H=H_0+\epsilon H_1\left(I_1,\ldots,I_N;\theta_1,\ldots,\theta_N\right),$$

[5]In principle, an integral system can never be consider ergodic, since the existence of invariant tori makes the system not metrically transitive. Nevertheless, if one limits to consider only the motion on a torus (and so all the ensemble averages) then the motion can be ergodic on it if the fundamental frequencies ω_j are not commensurate (i.e. if all the ratios $\frac{\omega_j}{\omega_k}$ are irrarational). In the case of commensurate frequencies, then the total motion is periodic on the torus and is not ergodic.

do any integral of motion exist for H? Suppose that the answer is yes, then let us call them F_j. The putative new integrals of motion should satisfy the relations

$$\{H, F_j\} = 0,\ \{F_i, F_j\} = \delta_{i,j}.$$

We assume that F_j is obtained from I_j by the perturbation ϵ, so we can expand it as

$$F_j = I_j + \epsilon F_j^{(1)} + \epsilon^2 F_j^{(2)} + \dots,$$

the same can be done for the Poisson parentheses and, up to the first order one should have

$$\{H, F_j\} \simeq \left\{H_0 + \epsilon H_1, I_j + \epsilon F_j^{(1)}\right\} \simeq \epsilon\left(\{H_1, I_j\} + \left\{H_0, F_j^{(1)}\right\}\right) = 0,$$

meaning that

$$\left\{H_0, F_j^{(1)}\right\} = -\{H_1, I_j\}.$$

In the hypotesis of motion confined in a bounded volume of the phace space, we can assume periodic bundary conditions and expand the functions in Fourier series

$$F_j^{(1)} = \sum_{\underline{m}} c_{\underline{m}}^{(j)} e^{i\underline{m}\cdot\underline{\theta}}.$$

Let us expand in Fourier series also the perturbation Hamiltonian

$$H_1 = \sum_{\underline{m}} g_{\underline{m}} e^{i\underline{m}\cdot\underline{\theta}},$$

so that the Poisson parentheses become

$$\begin{aligned}\left\{H_0, F_j^{(1)}\right\} &= \sum_{k=1}^{N}\left(\frac{\partial H_0}{\partial\theta_k}\frac{\partial F_j^{(1)}}{\partial I_k} - \frac{\partial F_j^{(1)}}{\partial\theta_k}\frac{\partial H_0}{\partial I_k}\right) = -\sum_{k=1}^{N}\frac{\partial F_j^{(1)}}{\partial\theta_k}\omega_k = \\ &= -i\sum_{k=1}^{N}\sum_{\underline{m}}\omega_k c_{\underline{m}}^{(j)} m_k e^{i\underline{m}\cdot\underline{\theta}},\end{aligned}$$

and

$$\begin{aligned}\{H_1, I_j\} &= \sum_{k=1}^{N}\left(\frac{\partial H_1}{\partial\theta_k}\frac{\partial I_j}{\partial I_k} - \frac{\partial I_j}{\partial\theta_k}\frac{\partial H_1}{\partial I_k}\right) = \frac{\partial H_1}{\partial\theta_j} = \\ &= i\sum_{\underline{m}} g_{\underline{m}} m_k e^{i\underline{m}\cdot\underline{\theta}},\end{aligned}$$

leading to the condition

$$c_{\underline{m}}^{(j)}\sum_{k=1}^{N}\omega_k m_k = g_{\underline{m}} m_k.$$

This means that relative integrals exist only if

$$c_{\underline{m}}^{(j)} = \frac{g_{\underline{m}} m_k}{\sum_{k=1}^{N}\omega_k m_k}.$$

The denominator is zero if the frequencies are commensurate or, in general, can assume values arbitrarily small. This means that so the coefficient $c_{\underline{m}}$ diverges and is not possible to find a well defined F_j.

To conclude, integrable systems are an exception! As soon as one adds a small perturbation the integrability disappears.

But this is also true for the ergodicity? Adding a small perturbation to an integrable system does it become ergodic? The answer is no: most of the invariant tori remain also after the perturbation is added as it has been shown by Kolmogorv, Arnold and Moser in the very important (KAM) theorem

Theorem 0.0.5. ***KAM (Kolmogorov, Arnold, Moser)*** *Given a perturbed Hamiltonian* $H = H_0(I_1, \ldots, I_N) + \epsilon H_1(I_1, \ldots, I_N; \theta_1, \ldots, \theta_N)$*, with* H_0 *sufficiently regular and*

$$\det \left| \frac{\partial^2 H_0}{\partial I_i \partial I_j} \right| \neq 0,$$

for $\epsilon < \epsilon^*$ *most of the invariant tori on the constant energy surface survive (**KAM tori**) to the perturbation and their measure tends to* 1 *for* $\epsilon \to 0$*.*

The KAM theorem show that periodic behavior can persists in nonintegrable system and nonergodicity is more robust that one can expect. A very evident of it is the motion of the Earth around the sun, which remains baiscally periodic despite the disturbance of the Moon an the other planets of the solar system.