Free energy & Entropy in Mean Field Ising

$$\frac{\Psi[m]}{N} = \frac{1}{2} J z m^2 - \frac{1}{\beta} \ln 2\,\mathrm{ch}\left(\beta(Jzm+h)\right)$$

Now the internal energy can be written in MF as

$$U/N = -\frac{J}{2} z m^2 - hm$$

and from C-W

$$h = \frac{1}{\beta} \mathrm{tgh}^{-1}(m) - Jzm \Rightarrow Jzm + h = \frac{1}{\beta} \mathrm{th}^{-1}(m)$$

$$\frac{\Psi(m)}{N} = \frac{U}{N} + \frac{J}{2} z m^2 + hm + \frac{Jz}{2} m^2 - \frac{1}{\beta} \ln 2\mathrm{ch}(\mathrm{th}^{-1}(m))$$

$$\frac{\Psi(m)}{N} = \frac{U}{N} + (Jzm+h)m - \frac{1}{\beta} \ln 2\mathrm{ch}(\mathrm{th}^{-1}(m))$$

$$\frac{\Psi(m)}{N} = \frac{U}{N} + \underbrace{\frac{m}{\beta} \mathrm{th}^{-1}(m) - \frac{1}{\beta} \ln 2\,\mathrm{ch}(\mathrm{th}^{-1}(m))}_{\frac{1}{2\beta}\left\{(1+m)\ln\frac{1+m}{2} + (1-m)\ln\frac{1-m}{2}\right\}}$$

$$\frac{m}{\beta}\,\mathrm{th}^{-1}(m) - \frac{1}{\beta}\ln 2\,\mathrm{ch}(\mathrm{th}^{-1}(m)) =$$

$$= \frac{1}{\beta}\left\{ mX - \ln(e^{X} + e^{-X}) \right\} \qquad X = \frac{1}{2}\ln\frac{1+m}{1-m}$$

$$\ln(e^{X}+e^{-X}) = X + \ln(1+e^{-2X}) = \qquad e^{-2X} = e^{-\ln\left(\frac{1+m}{1-m}\right)}$$

$$= X + \ln\left(1 + \frac{1-m}{1+m}\right) = X + \ln\frac{1+m+1-m}{1+m}$$

$$\frac{1}{\beta}\left\{ mX - \ln(e^{X}+e^{-X}) \right\} = \frac{1}{\beta}\left\{ (m-1)X - \ln\frac{2}{1+m} \right\}$$

$$= \frac{1}{\beta}\left\{ (m-1)\frac{1}{2}\ln\frac{1+m}{1-m} - \ln\frac{2}{1+m} \right\} =$$

$$= \frac{1}{2\beta}\left\{ (1-m)\ln\frac{1-m\,2}{1+m\,2} + 2\ln\frac{1+m}{2} \right\} =$$

$$= \frac{1}{2\beta}\left\{ (1-m)\ln\frac{1-m}{2} - (1-m)\ln\frac{1+m}{2} + 2\ln\frac{1+m}{2} \right\}$$

$$m - 1 + 2$$

$$\frac{\Psi(m)}{N} = \frac{U}{N} - \frac{TS}{N}$$

$$\frac{U}{N}(m) = -\frac{J}{2}z\,m^2 - h(m)\,m$$

$$\frac{S}{N} = \underbrace{-k_B\left\{ (1+m)\ln\frac{1+m}{2} + (1-m)\ln\frac{1-m}{2} \right\}}_{\text{entropy of "repulsive" lattice gas}}$$

That is the entropic part is formaly that of L-G without interaction i.e. pure hard spheres