from Langevin to Fokker-Planck using Ito calculus

$$dx = a(x,t)\,dt + b(x,t)\,dW$$

using Ito calculus for a generic function of x & t

$$df(x,t) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + \underbrace{\cancel{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2}}_{\text{second order}} + \cancel{\frac{\partial^2 f}{\partial x \partial t}dx\,dt}$$

dx and dt are uncorrelated

$$(dx)^2 = (a\,dt + b\,dW)^2 = \underbrace{a^2\cancel{dt^2}}_{\text{second order}} + 2ab\,\cancel{dt}\,dW + b^2 dW^2$$

$(dW)^2 = dt$ on average but

$$\langle (dW)^4 \rangle - dt^2 = \langle g^4 dt^2 \rangle - dt^2 = 2\,dt^2$$

$$(dW)^2 = g^2 dt \qquad \langle (dW)^2 \rangle = \langle g^2 \rangle dt$$

$$dW = g\sqrt{\Delta t}$$

$$(dx)^2 = b^2 dt$$

$$df = \frac{\partial f}{\partial x}[a\,dt + b\,dW] + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}b^2 dt$$

$$df = \left[\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}\right]dt + b\frac{\partial f}{\partial x}dW$$

$$\langle df \rangle = \left[\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}\right]dt$$

notice that the usual assumption $(dW)^2 = dt$ is not needed here

$$\frac{\langle df \rangle}{dt} = \frac{d}{dt}\langle f \rangle = \int dx \frac{\partial P}{\partial t} f(x) \qquad \text{for a } t\text{-indep function}$$

$$= \int dx\, P(x,t)\left[\underset{(a)}{a\frac{\partial f}{\partial x}} + \underset{(b)}{\frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}}\right] \qquad f \text{ indep ont}$$

a) $\int df\, aP = aP\Big| - \int dx \left[\frac{\partial}{\partial x}aP\right] f(x)$

b) $\int dx \frac{b^2}{2}P\frac{\partial}{\partial x}f' =$

$$= \int df' \frac{b^2}{2}P = f'\frac{b^2}{2}P\Big| - \int dx\left[\frac{\partial}{\partial x}\frac{b^2}{2}P\right]f'$$

$$= f'\frac{b^2}{2}P\Big|_{-\infty}^{+\infty} - \int df \frac{\partial}{\partial x}\frac{b^2}{2}P = f'\frac{b^2}{2}P\Big|_{-\infty}^{+\infty} - \left[\frac{\partial}{\partial x}\frac{b^2}{2}P\right]f\Big|_{\infty}^{+\infty}$$

$$+ \int dx\, f(x)\left[\frac{\partial^2}{\partial x^2}\frac{b^2}{2}P\right]$$

Assuming vanishing of all finite terms we get

$$\int dx\, P(x,t)\left[a\frac{\partial f}{\partial x}+\frac{b^2}{2}\frac{\partial^2}{\partial x^2}f\right]=$$

$$-\int dx\, f(x)\left[\frac{\partial}{\partial x}aP\right]+\int dx\, f(x)\,\frac{\partial^2}{\partial x^2}\frac{b^2}{2}P$$

thus for any $f(x)$ then

$$\boxed{\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}aP+\frac{1}{2}\frac{\partial^2}{\partial x^2}b^2P}$$

Fokker Planck

Ito calculus revised

$$df=\left(\frac{\partial f}{\partial t}+a\frac{\partial f}{\partial x}\right)dt+\underbrace{b\frac{\partial f}{\partial x}dW}_{O(\sqrt{dt})}+\frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}(dW)^2$$

$$(dW)^2=(dW)^2-dt+dt$$

$$df=\left(\frac{\partial f}{\partial t}+a\frac{\partial f}{\partial x}\right)dt+\underbrace{b\frac{\partial f}{\partial x}dW}_{\sqrt{dt}}+\underbrace{\boxed{\frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}\left[(dW)^2-dt\right]}}_{dt}$$

$$+\frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}dt$$