The Kramers Equation

From H. Risken "The Fokker-Planck Equation"

The Kramers eq. can be obtained from stochastic version of Hamilton's equations for a particle in a thermal bath

$$\frac{dx}{dt} = \frac{p}{M}$$

$$\frac{dp}{dt} = F(x) - \gamma p + \xi_e$$

with $\langle \xi_e(t) \rangle = 0$ and $\langle \xi(t)\, \xi_e(t') \rangle = \frac{2M\gamma}{\beta} \delta(t-t')$

These can be seen as a multi-dimensional F-P equation with additive noise

$$\frac{\partial P(\underline{x}, t)}{\partial t} = - \sum_i \frac{\partial}{\partial x_i} f_i(\underline{x})\, P + \frac{1}{2} \sum_i \epsilon_i \frac{\partial^2}{\partial x_i^2} P(\underline{x}, t)$$

valid for a multi-dimensional Langevin equation with additive noise

$$\dot{x}_i = f_i(\underline{x}) + \epsilon_i \eta_i(t) \qquad \langle \eta_i(t) \rangle = 0 \qquad \langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij} \delta(t-t')$$

in our case
$$\begin{cases} f_1 = \frac{p}{M} & \epsilon_1 = 0 \\ f_2 = F - \gamma p & \epsilon_2 = \frac{2M\gamma}{\beta} \end{cases}$$

$$\frac{\partial P(x p t)}{\partial p} = -\frac{\partial}{\partial x} f_1 P - \frac{\partial}{\partial p} f_2 P + \frac{\epsilon_2}{2} \frac{\partial^2}{\partial p^2} P$$

$$\frac{\partial P}{\partial p} = \underbrace{-\frac{p}{M}\frac{\partial P}{\partial x} - F\frac{\partial}{\partial p}P}_{\frac{\partial H}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial P}{\partial x} = \{H, P\}} + \gamma \frac{\partial}{\partial p} pP + \frac{M\gamma}{\beta}\frac{\partial^2}{\partial p^2} P$$

$$H = \frac{p^2}{2M} + V(x)$$

$$F(x) = -\frac{\partial V(x)}{\partial x}$$

$$\boxed{\frac{\partial P(x p t)}{\partial t} + \{P, H\} = \gamma \left\{ \frac{\partial}{\partial p} p P(x, p t) + M k_B T \frac{\partial^2}{\partial p^2} P(x p t) \right\}}$$

Kramers Equation

when $\gamma = 0$ the evolution of $P(x, p t)$ is Liouvillian

A Stationary equilibrium distribution can be obtained as

$$P(x, p) \propto e^{-\beta H(x p)}$$

since in this case $\{P, H\} = 0$ and in the r.h.s. the first and the second terms cancels

$$\frac{\partial}{\partial p} p e^{-\beta H}$$

$$\frac{M}{\beta}\frac{\partial^2}{\partial p^2} e^{-\beta H} = \frac{\cancel{M}}{\cancel{\beta}}\left(\frac{\partial}{\partial p}(-\cancel{\beta})\frac{p}{\cancel{M}} e^{-\beta H}\right) = -\frac{\partial}{\partial p} p e^{-\beta H}$$

For homogeneous system one can integrate the position defining the momentum distribution function

$$f(pt) = \int dx\, P(x p t)$$

$$\int dx \left\{ \frac{\partial P}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial H}{\partial x} \right\} = \frac{p}{M}\int dx \frac{\partial P}{\partial x} + \int dx\, \underbrace{\frac{\partial P}{\partial x} F(x)}_{F \neq 0 \text{ only at boundaries}}$$

$$P(x,pt)\Big|_{x=-\infty}^{x=+\infty} = 0$$

then

$$\frac{\partial f(pt)}{\partial t} = \gamma \underbrace{\left\{ \frac{\partial}{\partial p} p f(pt) + \frac{M}{\beta}\frac{\partial^2}{\partial p^2} f(pt) \right\}}_{\text{collision term}}$$

Contrary to Boltzmann's eq the collision term is local in momenta

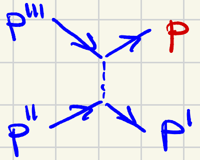
$$\frac{\partial f(pt)}{\partial t} = \int dp' dp'' dp'''\, W_{pp' \to p''p'''} \left\{ f(p'') f(p''') - f(p) f(p') \right\}$$

and does not depend non linearly from the f itself

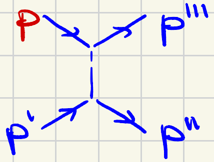
The Boltzmann's Eq is indeed a Master Equation of the kind

$$\frac{\partial P(x,t)}{\partial t} = \int dx' \, W(x|x') P(x' t) - \int dx' \, W(x'|x) P(x,t)$$

where the first terms comes from (IN) processes



and the second from (OUT) processes



There is a link btw Master Eq and Fokker-Planck which is expressed by the Kramers-Moyal expansion

$$\int dx' \{ W(x|x') P(x' t) - W(x|x') P(x t) \} =$$

$$= \sum_{1}^{\infty} {}_n \frac{(-1)^m}{m!} \left[\frac{\partial^m}{\partial x^m} \alpha_m(x) P(x,t) \right]$$

$$\alpha_n(x) = \int dx' (x'-x)^n W(x'|x)$$

and Fokker-Planck is obtained

selecting α_n with $n = 1, 2$