Brownian particle in absence or presence of external force

The Langevin equation in absence of ext force

$$\dot{p} = -\gamma p + \xi_e \qquad \langle \xi_e(t) \rangle = 0 \qquad \langle \xi(t)\xi(t') \rangle = \frac{2M\gamma}{\beta}\delta(t-t')$$

dissipation

fluctuations in random force

both are due to interaction with the bath

p is a Ornstein Uhlenbeck process

$$p(t) = e^{-\gamma t} p(0) + \int_0^t dt' \, e^{-\gamma(t-t')} \xi_e(t')$$

the average relaxes $\to 0$ on a time scale γ^{-1}

$$\langle p(t) \rangle = e^{-\gamma t} p(0) \qquad \forall \, p(0)$$

the variance approach asymptotically the Maxwell-Boltzmann's prediction

$$\langle p^2(t) \rangle = e^{-2\gamma t} p^2(0) + \int_0^t dt' \int_0^t dt'' e^{-\gamma(2t-t'-t'')} \langle \xi(t')\xi(t'') \rangle$$

$+$ vanishing terms which contains $\langle \xi(t') \rangle = 0$

using the δ correlation of the white noise ξ

$$\langle p^2(t)\rangle = e^{-2\gamma t}\, p^2(0) + \frac{2M\cancel{\gamma}}{\beta}\, e^{-2\gamma t}\, \frac{e^{2\gamma t}-1}{2\cancel{\gamma}}$$

cancellation occurs because
damping γ is related with the
"variance" of the noise ξ

for $\gamma t \gg 1 \quad t \gg \gamma^{-1}$

$$\langle p^2(t)\rangle \longrightarrow \frac{M}{\beta} \qquad \text{cfr Maxwell Boltzmann} \quad \rho(p) \propto e^{-\beta \frac{p^2}{2m}}$$

equilibrium property
do not depend on γ

Drift under external constant force

$$\dot{p} = -\gamma p + \xi + F$$

$$p(t) = e^{-\gamma t} p(0) + \int_0^t dt' e^{-\gamma(t-t')} \left(\xi(t) + F \right)$$

now on average

$$\langle p(t) \rangle = e^{-\gamma t} p(0) + F e^{-\gamma t} \frac{e^{\gamma t} - 1}{\gamma}$$

for $\gamma t \gg 1$ $\quad \langle p(t) \rangle = \frac{F}{\gamma}$ Aristotele's law

for example for a charged particle in uniform external field $F_x = eE_x$ we have a constant current density for n independent particles $J_x = ne \langle v_x \rangle$ for $\gamma t \gg 1$

$$J_x = \frac{ne}{M} \frac{e E_x}{\gamma}$$

conductivity

the Drude model

$$\boxed{\sigma = n \frac{e^2}{M\gamma} = n \frac{e^2 \tau}{M}}$$

depends on γ

The mobility is a single particle quantity

$$\langle \vec{v} \rangle = \mu \vec{E}$$

mobility in isotropic media under the action of E

$$\langle v_x \rangle = \frac{eE_x}{M\gamma} \qquad \mu = \frac{e}{M\gamma}$$

depends on γ

Mobility in the presence of external forces are related to diffusional process in absence of any external force

Diffusion of Brownian particle in absence of any external force

evaluate the mean square displacement of a particle from its initial position $(d=1)$

$$\langle |X(t)-X(0)|^2 \rangle \xrightarrow[\gamma t \gg 1]{} 2Dt$$

diffusion coefficient

Notice that for a ballistic particle

$X(t) = X(0) + v(0)t$ and there fore

$(X(t)-X(0))^2 = v^2(0)\, t^2$ which indeed is a signature of time reversal motion

Evaluate $\langle |X(t)-X(0)|^2 \rangle$ in the $\gamma t \gg 1$ by consider the overdamped limit $\gamma t \gg 1$ formally

$$\gamma e^{-\gamma(t-t')} \longrightarrow \delta(t-t') \quad (t>t')$$

$$p(t) = \overbrace{e^{-\gamma t} p(0)}^{0} + \frac{1}{\gamma}\gamma\int_0^t dt'\, e^{-\gamma(t-t')}\left(\xi(t)+F\right)$$

$$p(t) = \frac{\xi(t)+F}{\gamma} = M\dot{x}$$

dissipation

overdamped limit

$$\dot{x} = \frac{1}{M\gamma}\left[F+\xi\right]$$

when $F=0$

$$X(t)-X(0) = \frac{1}{M\gamma} \underbrace{\int_0^t dt'\, \xi(t')}_{\text{the Wiener Process}}$$

$$\langle (X(t)-X(0))^2 \rangle = \frac{1}{M^2\gamma^2} \int_0^t dt_1 \int_0^t dt_2 \langle \xi(t_1)\xi(t_2) \rangle$$

$$= \frac{1}{M^2\gamma^2} \frac{2M\gamma}{\beta} t$$

fluctuation

dissipation

which is a diffusive beavior with $D = \dfrac{1}{M\gamma\beta}$

Drift under external force (electrom.) in contact with dissipative bath

$$\mu = \frac{e}{M\gamma}$$

Diffusion in dissipative environment

$$D = \frac{k_B T}{M\gamma}$$

diffusion constant

$$\mu = \frac{eD}{k_B T}$$

electrical mobility

Einstein Relation

Response to external time varying force

$$p(t) = e^{-\gamma t} p(0) + \int_0^t dt' \, e^{-\gamma(t-t')} \left(\xi_e(t) + F(t) \right)$$

$$\langle p(t) \rangle = e^{-\gamma t} p(0) + \int_0^t dt' \, e^{-\gamma(t-t')} F(t')$$

Consider a sinusoidal external force e.g.

$$F(t) = F_0 e^{i\omega t}$$

$$\int_0^t dt' \, e^{\gamma t'} e^{i\omega t'} = \frac{e^{(\gamma+i\omega)t} - 1}{\gamma + i\omega}$$

for $\gamma t \gg 1$

$$\langle p(t) \rangle \simeq F_0 \frac{e^{i\omega t}}{\gamma + i\omega} = \frac{F(t)}{\gamma + i\omega}$$

for forces of electrical origin this leads to the complex conductivity Drude relation

$$\sigma = \frac{me^2}{M(\gamma + i\omega)}$$

$$\boxed{\mathrm{Re}\,\sigma = \frac{n e^2}{M(\gamma^2 + \omega^2)} \gamma}$$

Lorentzian absorption

Generalised response to external time-dependent field

generalised Langevin eq.

$$\dot{p} = -\int_{-\infty}^{t} dt' \underbrace{\mathcal{M}(t-t')}_{\text{the Memory Friction Kernel}} p(t') + \xi(t) + F(t)$$

Fourier transform

$$i\omega\, p(\omega) = -\mathcal{M}(\omega)\, p(\omega) + \xi(\omega) + F(\omega)$$

take average

$$\langle p(\omega) \rangle = \frac{F(\omega)}{i\omega + \mathcal{M}(\omega)}$$

and the generalised ω dependent conductivity is

$$\sigma(\omega) = \frac{1}{i\omega + \mathcal{M}(\omega)}$$

memory function is the F-transform of the memory-friction kernel