Example of Homogenous linear Response

small variation of homogenous chemical potential in the grand-canonical ensemble

$H_0 \rightarrow \hat{H} - \mu \hat{N}$

$V \rightarrow -\delta\mu \hat{N}$

for a generic $\hat{A}$

$$\langle \hat{A} \rangle_{\delta\mu} - \langle \hat{A} \rangle_0 = \delta\mu \int_0^\beta d\tau \left[\langle \hat{N}_I(\tau)\, \hat{A}(0) \rangle_0 - \langle \hat{N}_I(\tau) \rangle_0 \langle \hat{A}(0) \rangle_0 \right]$$

now 0 is the system without perturbation therefore

$$\frac{\partial \langle \hat{A} \rangle}{\partial \mu} = \int d\tau \left[\langle \hat{N}_I(\tau) \hat{A} \rangle_0 - \langle \hat{N}_I(\tau) \rangle_0 \langle \hat{A} \rangle_0 \right]$$

but for global gauge invariance (number conservation)

$$N_I = e^{\tau(H-\mu N)} \underbrace{\hat{N}\, e^{-\tau(H-\mu N}}_{\text{commutes}} = \hat{N} \text{ indep } \tau!$$

$$\boxed{\frac{\partial \langle \hat{A} \rangle}{\partial \mu} = \beta \left[\langle \hat{N} \hat{A} \rangle - \langle \hat{N} \rangle \langle \hat{A} \rangle \right]}$$

for example let $\hat{A} = \hat{H}$ and calculate the derivative of internal energy ...

let $\hat{A} = \hat{N}$

$$\frac{\partial \langle \hat{N} \rangle}{\partial \mu} = \beta [\underbrace{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}]$$

fluctuations are not superextensive!

response to a variation of μ

let $\hat{A} = \hat{H}$ and H is a perfect gas

$$\frac{\partial \langle \hat{H} \rangle}{\partial \mu} = \beta [\langle \hat{N} \hat{H} \rangle - \langle \hat{N} \rangle \langle \hat{H} \rangle]$$

$$\hat{N} = \Sigma_\alpha \underbrace{a^+_\alpha a_\alpha}_{\hat{n}_\alpha} \qquad \hat{H} = \Sigma_\alpha \varepsilon_\alpha \underbrace{a^+_\alpha a_\alpha}_{\hat{n}_\alpha}$$

$$\langle \hat{N} \hat{H} \rangle = \sum_{\alpha\beta} \varepsilon_\beta \langle \hat{n}_\alpha \hat{n}_\beta \rangle$$

but different "modes" α are independent

$$\langle \hat{N} \hat{H} \rangle = \Sigma_\alpha \varepsilon_\alpha \langle \hat{n}_\alpha^2 \rangle$$

$$\frac{\partial \langle H \rangle}{\partial \mu} = \beta \Sigma_\alpha \varepsilon_\alpha (\langle n_\alpha^2 \rangle - \langle n_\alpha \rangle^2)$$

$$\cancel{\langle n_\alpha^2 \rangle = \frac{1}{(e^{\beta(\varepsilon_\alpha - \mu)} - 1)^2}}$$

$$\langle n_\alpha \rangle = \frac{\partial \Omega}{\partial \varepsilon_\alpha} = -\frac{1}{\beta} \frac{\partial}{\partial \varepsilon_\alpha} \ln \mathrm{tr}\, e^{-\beta [\Sigma_\alpha \varepsilon_\alpha \hat{n}_\alpha - \mu \Sigma_\alpha \hat{n}_\alpha]}$$

$$-\beta [\langle n_\alpha^2 \rangle \cancel{= \frac{\partial^2 \Omega}{\partial \varepsilon_\alpha^2}} - \langle n_\alpha \rangle^2] = \frac{\partial^2 \Omega}{\partial \varepsilon_\alpha} = \frac{\partial \langle n_\alpha \rangle}{\partial \varepsilon_\alpha}$$

$$\Omega = \mp \frac{1}{\beta} \Sigma_\alpha \ln \left(1 \pm e^{-\beta(\varepsilon_\alpha - \mu)}\right)$$

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$$\langle n_\alpha \rangle = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} \pm 1}$$

$$\frac{\partial \langle n_\alpha \rangle}{\partial \varepsilon_\alpha} = - \frac{\beta e^{\beta(\varepsilon_\alpha - \mu)}}{\left(e^{\beta(\varepsilon_\alpha - \mu)} \pm 1\right)^2}$$

$$\langle n_\alpha^2 \rangle - \langle n_\alpha \rangle^2 = \frac{e^{\beta(\varepsilon_\alpha - \mu)}}{\left(e^{\beta(\varepsilon_\alpha - \mu)} \pm 1\right)^2}$$

thus for a quantum gas

$$\frac{\partial \langle \hat{N} \rangle}{\partial \mu} = \beta \Sigma_\alpha \frac{e^{\beta(\varepsilon_\alpha - \mu)}}{\left(e^{\beta(\varepsilon_\alpha - \mu)} \pm 1\right)^2}$$

$$\frac{\partial \langle \hat{H} \rangle}{\partial \mu} = \beta \Sigma_\alpha \varepsilon_\alpha \frac{e^{\beta(\varepsilon_\alpha - \mu)}}{\left(e^{\beta(\varepsilon_\alpha - \mu)} \pm 1\right)^2}$$