Bose-Einstein Condensation

the average # occupation number in the Bose statistics

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}$$

shows a divergence when $\varepsilon = \mu$ however we see that if

$$\mu \leq \min_{\alpha} \varepsilon_{\alpha}$$

for some kinds of dispersions equality sign can occur as well.

Consider a general case of

$$\varepsilon(\vec{k}) = a k^{b}$$

$b=2$ massive non relativistic

$b=1$ massless ultrarelativistic

then the associate DoS

$$\mathcal{N}(\varepsilon) \propto \varepsilon^{3/b - 1} \quad \text{in } d=3 \text{ (spatial dimensions)}$$

$$\mathcal{N}(\varepsilon) \propto \varepsilon^{\frac{d}{b} - 1} \quad \text{in } d \text{ dimensions}$$

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therefore the integral appearing in the definition of $\langle N \rangle$ ②

$$\langle N \rangle = \int d\epsilon\, N(\epsilon)\, \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$$

can be finite as $\mu = 0$ if the divergence $\varepsilon \simeq 0$

$$\int_0 d\epsilon\, \epsilon^{\frac{d}{b}-1} \frac{1}{1+\beta\varepsilon-1} = \int_0 d\epsilon\, \frac{\epsilon^{\frac{d}{b}-2}}{\beta} = \frac{\text{const}}{\beta^{d/b}} \quad (*)$$

can be integrated i.e. $\frac{d}{b}-2 > 1$ $\boxed{d > b}$

for massive non relativistic particles $b=2$

$d > 2$

for massless ultra relativistic particles

$d > 1$

note that the black body radiation follows the massless ultra relativistic regime but in that case $\mu = 0$ independent on T!

Therefore this conditions cannot be accomplished in 1-d case ...

(*) If the integral exists for $\mu = 0$ then

$$c\int d\epsilon\, \varepsilon^{\frac{d}{b}-1} \frac{1}{e^{\beta\varepsilon}-1} = \frac{c}{\beta^{\alpha+1}} \int_0^\infty dx\, \frac{x^\alpha}{e^x-1}$$

If integral exists then we can qualitatively (3)
address the $\langle N \rangle$ as a function of μ

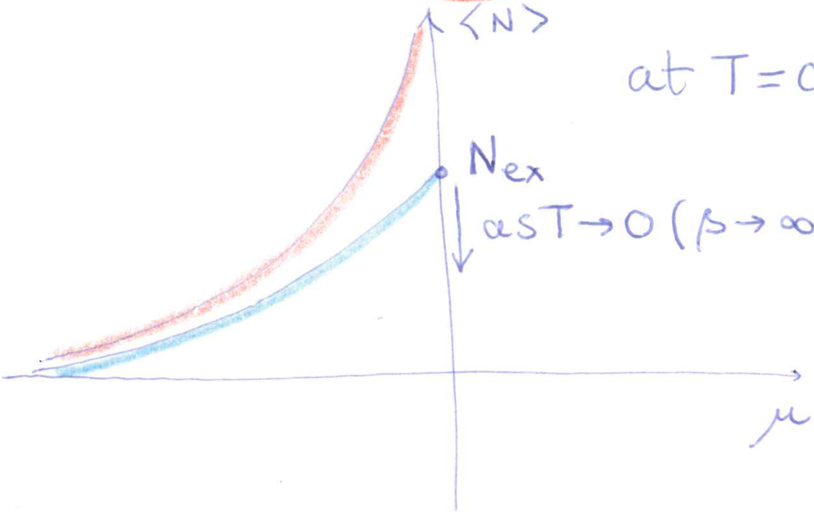
for $\mu \to -\infty$ $z \to 0$

$\langle N \rangle \simeq e^{\beta\mu} \int d\varepsilon\, N(\varepsilon)\, e^{-\beta \varepsilon}$ exponentially small in μ

$\langle N \rangle \to \frac{\text{const}}{\beta}$ $\mu \to 0$ $d > 2$

$\langle N \rangle \to$ diverge $\mu \to 0$ $d \leq 2$

} massive non relativistic

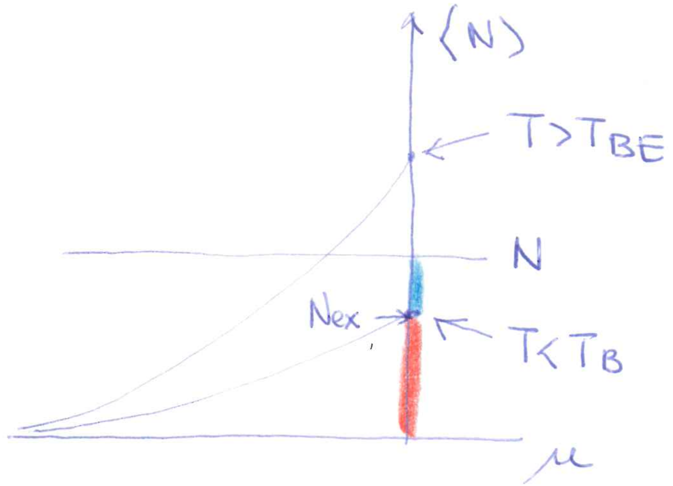


at $T = \text{const}$

as $T \to 0$ $(\beta \to \infty)$ $\sim \frac{1}{\beta^{\frac{d}{2}}}$ massive non relativistic

However to invert the function $\langle N \rangle(\mu)$ one is forced to exclude $\mu = 0$ if $d > 2$ since we want to solve the equation

$$\langle N \rangle(\mu) = N$$ ← actual # of particle



there exist a temperature below which

$N = \langle N \rangle(\mu)$

has no solution

$$N = N_0 + N_{ex}$$

therefore we must resort to the discrete representation of the total average number

$$\langle N \rangle = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle = \sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

an substract the $\vec{k} = 0$ state from the sum when $\mu = 0$

$$\langle N \rangle = \frac{1}{e^{\beta(\varepsilon_{k=0})} - 1}^{*} + \sum_{\vec{k}} \frac{1}{e^{\beta \varepsilon_k} - 1}$$

# of particles in excited states

# of particles in Ground State

this can be aproximated with an integral

The Bose-Einstein temperature is defined by

$N_0 = 0$

$N = N_{ex}(T_{BE})$ but $N_{ex}(T) = \frac{C}{\beta^{\alpha+1}}$

massive non relativ

$\alpha = \frac{d}{2} - 1$

$$\frac{N_{ex}(T)}{N} = \left(\frac{k_B T}{k_B T_{BE}}\right)^{\alpha+1}$$

* N_0 is extensive if $\mu = \varepsilon_{k=0} - \delta\mu$ $\quad \delta\mu = \frac{1}{\beta} \ln\left(1 + \frac{1}{N_0}\right)$

$$\boxed{N_{ex}(T) = N\left(\frac{T}{T_{BE}}\right)^{\alpha+1}}$$

but for number conservation

$$N = N_0(T) + N\left(\frac{T}{T_{BE}}\right)^{\alpha+1}$$

therefore

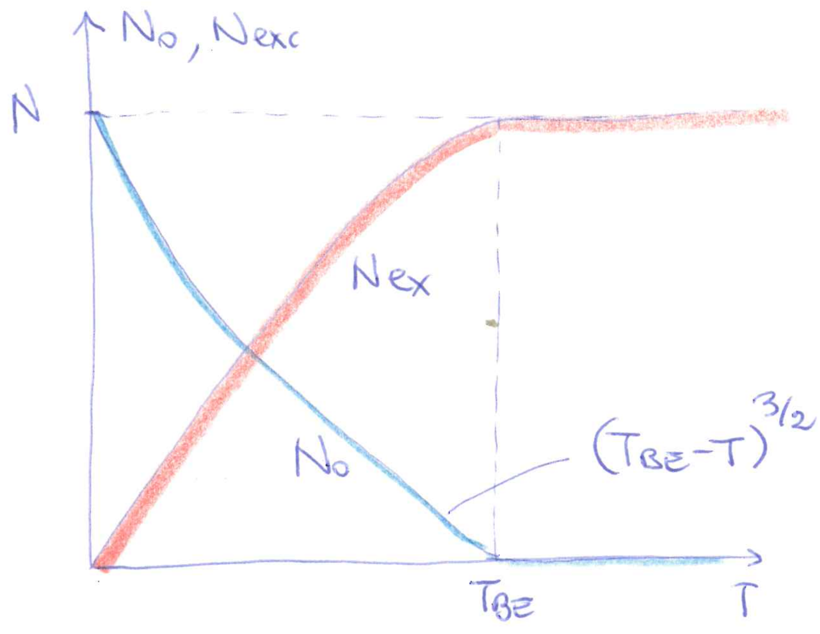
$$\boxed{N_0(T) = N\left(1 - \left(\frac{T}{T_{BE}}\right)^{\alpha+1}\right)}$$

for massive non relativistic particles in $d=3$

$$\mathcal{N}(\varepsilon) = \frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} g \, \varepsilon^{1/2}$$

spin degeneracy

$$N_{exc}(T) = g\frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}(k_B T)^{3/2} \underbrace{\int_0^\infty dx \, \frac{\sqrt{x}}{e^x - 1}}_{\frac{\sqrt{\pi}}{2}\underbrace{\zeta(3/2)}_{2,612}} \qquad T \leq T_{BE}$$



$$N_{ex} = g \frac{V}{(2\pi)^2} \left(\frac{2m k_B T}{h^2 (2\pi)^2} \right)^{3/2} \frac{\sqrt{\pi}}{2} \, \zeta$$

$$g V \left(\frac{1}{\cancel{(2\pi)}^{4/3}} \frac{2m k_B T}{\frac{h^2}{\cancel{(2\pi)^2}}} \frac{\pi^{\cancel{1}/3}}{\cancel{2^{2/3}}} \right)^{3/2} \zeta$$

$\pi^{2/3}$ $\cancel{\pi^{2/3}}$ $\frac{1}{2} \frac{2}{3}$

$N_{ex} = g V / \lambda^3(T) \, \zeta$ massive non relativistic $d=3$

the T_{BE} is defined by

$$N = N_{ex}(T_{BE}) = \frac{g V}{\lambda^3(T_{BE})} 2{,}612 \qquad \frac{N}{V} = \rho = \frac{1}{e^3}$$

$$\boxed{\left(\frac{\lambda_{BE}}{e} \right)^3 = g \; 2{,}612}$$

when $g = 1$ (by expressing λ)

$$k_B T_{BE} = \frac{\hbar^2}{2m} 4\pi \left(\frac{\rho}{2{,}612} \right)^{2/3}$$ depends on density

for Helium (isotope He_2)

$T_{BE} = 3{,}1 \, °K$

$T_{BE}^{exp} = 2{,}19 \leftarrow$ Superfluid + Interactions (short range repulsive)

(6)

$$\langle N_0\rangle = \frac{1}{e^{\beta(\varepsilon_0-\mu)}-1} \qquad e^{\beta(\varepsilon_0-1)}-1 = \frac{1}{\langle N_0\rangle}$$

$$\varepsilon_0-\mu = \frac{1}{\beta}\ln\left(1+\frac{1}{\langle N_0\rangle}\right) \simeq \frac{k_BT}{\langle N_0\rangle}$$

only in thermodynamic limit actually $\mu \to \varepsilon_0^-$

$$U = \int d\epsilon \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)}-1} N(\varepsilon)$$ if $\varepsilon_0 = 0$ the contributo

$$U = \frac{\varepsilon_0}{e^{\beta(\varepsilon_0-\mu)}-1} + \sum_{\alpha>0} \varepsilon_\alpha n_\alpha$$

can be omitted becouse of numerator

and as $T < T_{BE}$

$$U = g\frac{V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\varepsilon \frac{\varepsilon\sqrt{\varepsilon}}{e^{\beta\varepsilon}-1}$$ for 3d non relativistic particles

$\beta^{5/2}$

introducing the De Broglie W/L

$$U = \frac{3}{2}k_BT\frac{V}{\lambda^3}\zeta_{5/2} \qquad \zeta_{5/2} = \underbrace{\frac{4}{3\sqrt{\pi}}\int_0^\infty dx \frac{x^{3/2}}{e^x-1}}_{1,342}$$

A relation can be proved for ideal gas in 3d (7)

$PV = \frac{2}{3} U$ both classical & Quantum

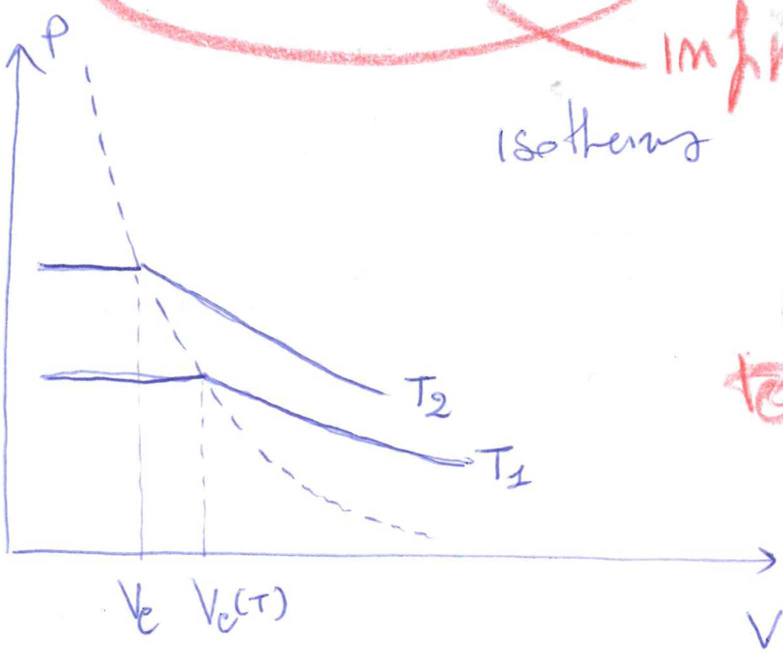
$P\cancel{V} = k_B T \frac{\cancel{V}}{\lambda^3} g_{5/2}$

it does not depend on volume (as in the black body)

$\Rightarrow \quad k_T = -\frac{1}{V}\frac{\partial V}{\partial P} \to \infty$ infinite susceptibility

↑

Huge response to density



for any $V < V_c$ ∂P

since $k_B T_{BE} = 4\pi \frac{\hbar^2}{2m}\left(\frac{\rho}{2{,}612}\right)^{2/3}$

at fixed temperatures varying the volume one can reach the critical volume as far as

$\left(\frac{\lambda(T)}{e}\right)^3 = g\, 2{,}612 \qquad e^3 = \frac{V_c}{N}$ ← critical value

$\frac{\lambda^3(T)\, V_c}{N} = g\, 2{,}612 \qquad V_c = \frac{g N\, 2{,}612}{\lambda^3(T)}$ ← fixed temp

BEC phase when $T < T_{BE}$ or when $V < V_c$