Bose & Fermi Gases

$$\mathcal{Z}_{gc} = \sum_{\{n_\alpha\}} \prod_\alpha e^{-\beta(\varepsilon_\alpha-\mu)n_\alpha} = \prod_\alpha \sum_{n_\alpha} e^{-\beta(\varepsilon_\alpha-\mu)n_\alpha}$$

$$\ln \mathcal{Z}_{gc} = \sum_\alpha \ln \sum_{n_\alpha} e^{-\beta(\varepsilon_\alpha-\mu)n_\alpha} = \pm \sum_\alpha \ln\left(1 \pm e^{-\beta(\varepsilon_\alpha-\mu)}\right)$$

$$PV = \pm \frac{1}{\beta} \sum_\alpha \ln\left(1 \pm e^{-\beta(\varepsilon_\alpha-\mu)}\right)$$

after introducing the DOS

$$\boxed{PV = \pm \frac{1}{\beta} \int d\varepsilon\, \mathcal{N}(\varepsilon) \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right)}$$

to obtain the eq of state we need

$$\boxed{\langle N \rangle = \sum_\alpha \frac{1}{e^{\beta(\varepsilon_\alpha-\mu)} \pm 1} = \int d\varepsilon\, \mathcal{N}(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}}$$

another equation of interest:

$$\boxed{\langle H \rangle = U = \sum_\alpha \frac{\varepsilon_\alpha}{e^{\beta(\varepsilon_\alpha-1)} \pm 1} = \int d\varepsilon\, \mathcal{N}(\varepsilon)\, \varepsilon \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}}$$

It is interesting to note the the classical perfect gas can be obtaind as

$$\mathcal{Z}_{gc} = \sum_{\{n_\alpha\}} \prod_\alpha \frac{e^{-\beta(\varepsilon_\alpha-\mu)n_\alpha}}{n_\alpha!} = \prod_\alpha \sum_{n_\alpha} \frac{e^{-\beta(\varepsilon_\alpha-\mu)n_\alpha}}{n_\alpha!}$$

$$\mathcal{Z}_{gc} = \prod_\alpha \exp\left[e^{-\beta(\varepsilon_\alpha-\mu)}\right]$$

therefore

$$PV = \frac{1}{\beta}\sum_\alpha e^{-\beta(\varepsilon_\alpha-\mu)} = \frac{e^{\beta\mu}}{\beta}\sum_\alpha e^{-\beta\varepsilon_\alpha}$$

$$\langle N\rangle = -\frac{\partial\Omega}{\partial\mu} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln\mathcal{Z}_{gc} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\sum_\alpha e^{-\beta(\varepsilon_\alpha-\mu)} = \sum_\alpha e^{-\beta(\varepsilon_\alpha-\mu)}$$

$$\langle N\rangle = e^{\beta\mu}\sum_\alpha e^{-\beta\varepsilon_\alpha}$$

and

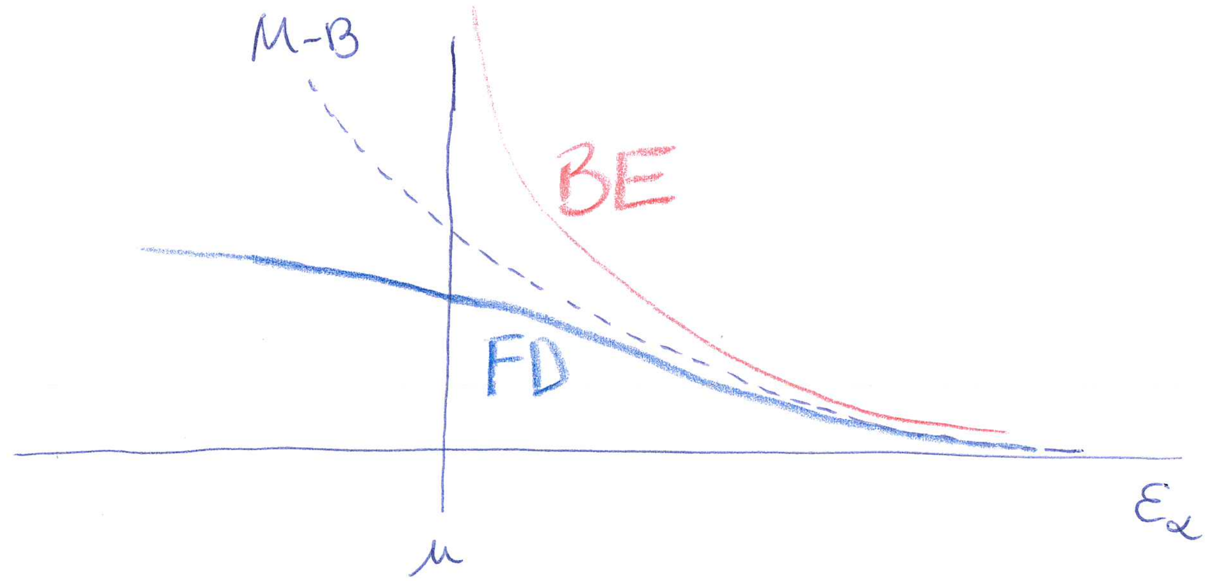
$$\frac{PV}{\langle N\rangle} = \frac{1}{\beta}$$ which is the perfect's gas law

we therefore can compare

Maxwell Boltzman $\langle n_k \rangle = e^{-\beta(\epsilon_k - \mu)}$

Fermi-Dirac $\langle n_k \rangle = \dfrac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$

Bose Einstein $\langle n_k \rangle = \dfrac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$



a general kind of DOS is the power-low

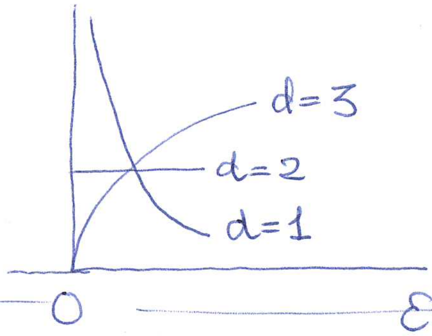
$$\mathcal{N}(\varepsilon) \propto \varepsilon^{\alpha}$$

for $\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2m}$ (non relativistic particle)

$d=3 \quad \alpha = 1/2$

$d=2 \quad \alpha = 0$

$d=1 \quad \alpha = -1/2$



An universal relation follows for a power-low DOS

for relativistic particles (massless)

$$\epsilon = c|\vec{p}|$$

$$\mathcal{N}(\varepsilon) \propto \varepsilon^{2} \qquad \alpha = 2$$

Between the product PV and U

$$\mathcal{N}(\varepsilon) = \mathcal{C}\,\varepsilon^{\alpha} = \frac{\mathcal{C}}{\alpha+1}\frac{d}{d\varepsilon}\varepsilon^{\alpha+1}$$

$$\int d\epsilon\, \mathcal{N}(\varepsilon) \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right) =$$

$$= \frac{\mathcal{C}}{\alpha+1}\int \frac{d\,\varepsilon^{\alpha+1}}{d\,\varepsilon} \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right) =$$

$$= \frac{\mathcal{C}}{\alpha+1}\left[\varepsilon^{\alpha+1}\ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right)\Big|_0^{\infty} - \int d\varepsilon\, \varepsilon^{\alpha+1}\frac{\mp\beta e^{-\beta(\varepsilon-\mu)}}{1 \pm e^{-\beta(\varepsilon-\mu)}}\right.$$

$\varepsilon^{\alpha+1} \ln(1 \pm e^{-\beta(\varepsilon-\mu)})\Big|_0^\infty$ (5)

assume that for Bosons $\mu < 0 \Rightarrow$ normal state $T > T_{BE}$

if $\alpha+1>0$ the term vanishes for $\varepsilon \to 0$

as $\varepsilon \to \infty$ $\quad \ln(1 \pm e^{-\beta(\varepsilon-\mu)}) \simeq \pm e^{-\beta(\varepsilon-\mu)}$

and converges to zero therefore under this assumptions

$$\int d\varepsilon \, \mathcal{N}(\varepsilon) \ln(1 \pm e^{-\beta(\varepsilon-\mu)}) = \pm \frac{\beta}{\alpha+1} \int d\varepsilon \, \mathcal{N}(\varepsilon) \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

and

$$PV = \frac{1}{\alpha+1} \int d\varepsilon \, \mathcal{N}(\varepsilon) \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

$$\boxed{PV = \frac{U}{\alpha+1}}$$

this relation hold also for Boltzmann particles eg the perfect classical gas has $\alpha = 1/2$ and

$$PV = \frac{2}{3} U = \frac{2}{3}\left(\frac{3}{2} N k_B T\right)$$

which gives the perfect gas law

The ground state

- fermions $\langle n_\alpha \rangle = \begin{cases} = 0 & \varepsilon > E_F \\ = 1 & \varepsilon < E_F \end{cases}$

$\lim_{T \to 0} \mu(T) = E_F$ the Fermi Energy

in this case

$$\langle N \rangle = \int_0^{E_F} d\varepsilon \, \mathcal{N}(\varepsilon)$$

$$U = \int_0^{E_F} d\varepsilon \, \mathcal{N}(\varepsilon) \, \varepsilon$$

for an universal $\mathcal{N}(\varepsilon) = \mathbb{C} \, \varepsilon^\alpha$

$$\langle N \rangle = \mathbb{C} \frac{E_F^{\alpha+1}}{\alpha+1}$$

$$\frac{U}{\langle N \rangle} = \frac{\alpha+1}{\alpha+2} E_F$$

$$U = \mathbb{C} \frac{E_F^{\alpha+2}}{\alpha+2}$$

$$PV = \mathbb{C} \frac{E_F^{\alpha+2}}{(\alpha+2)(\alpha+1)}$$

for massive nonrelativistic particles in 3d

$$\varepsilon(\vec{p}) = \frac{\vec{p}^{\,2}}{2m}$$

$$\mathcal{N}(\varepsilon) = \underbrace{g \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2}}_{\mathbb{C}} \sqrt{\varepsilon}$$

Spin degeneracy [illegible]

that is

$$\langle N \rangle = g \frac{V}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \frac{E_F^{3/2}}{3/2} = g \frac{V}{6\pi^2} \left(\frac{2m}{\hbar^2} E_F \right)^{3/2}$$

which gives

$$E_F = \frac{\hbar^2}{2m} \Big[\underbrace{\frac{6\pi^2}{g} \frac{\langle N \rangle}{V}}_{k_F^2} \Big]^{2/3} \propto \rho^{2/3}$$

$$\frac{PV}{\langle N \rangle} = \frac{E_F}{\alpha + 2} = \frac{2}{5} E_F$$

cf for classical gas $\frac{PV}{\langle N \rangle} = k_B \overset{\text{independent}}{T} \longrightarrow 0$ as $T \to 0$

the $T=0$ isotherms of the Fermi perfect gas can be derived by expressing E_F as a function of $\rho = \langle N \rangle / V$

$$\frac{P}{\rho} = \frac{E_F(\rho)}{\alpha + 2} \qquad P = \frac{\rho E_F(\rho)}{\alpha + 2}$$

for massive nonrelativistic particles

$$P = \frac{2}{5} \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{g} \rho \right)^{2/3} \rho$$

$$P \simeq \frac{\text{Const}}{V^{5/3}}$$