Virial Coefficient for the Bose and the Fermi gases

Fermi

Bose

$$PV = \pm \frac{1}{\beta}\int d\epsilon\, N(\varepsilon)\, \ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

$$\langle N \rangle = \int d\epsilon\, N(\varepsilon)\, \frac{z}{e^{\beta\epsilon} \pm z}$$

formally

$$\ln\left(1 \pm z e^{-\beta\epsilon}\right) = \begin{cases} \nearrow\ \sum_{1}^{\infty} {}_n (-1)^{n+1} \dfrac{z^n e^{-n\beta\epsilon}}{n} = -\sum_{1}^{\infty} {}_n (-1)^n \dfrac{z^n e^{-n\beta\epsilon}}{n} \\ \searrow\ -\sum_{1}^{\infty} {}_n \dfrac{z^n e^{-n\beta\epsilon}}{n} \end{cases}$$

$$\ln\left(1 \pm z e^{-\beta\epsilon}\right) = -\sum_{1}^{\infty} {}_n (\mp)^n \frac{z^n e^{-n\beta\epsilon}}{n}$$

$$\frac{z}{e^{\beta\epsilon} \pm z} = z e^{-\beta\epsilon} \frac{1}{1 \pm z e^{-\beta\epsilon}} = z e^{-\beta\epsilon} \sum_{0}^{\infty} {}_n (\mp)^n z^n e^{-n\beta\epsilon}$$

$$\beta \frac{PV}{\langle N \rangle} = \mp \frac{\sum_{1}^{\infty} {}_n (\mp)^n \dfrac{z^n Q_n}{n}}{z \sum_{0}^{\infty} {}_n (\mp)^n z^n Q_{n+1}} \qquad Q_n = \int d\epsilon\, e^{-\beta\epsilon} N(\varepsilon)$$

leading order is

$$\frac{\beta PV}{\langle N \rangle} = \mp \frac{\mp z Q_1}{z Q_1} = 1$$

$$\frac{\beta P V}{\langle N\rangle} = \mp \frac{\mp z Q_1 + \sum_{2}^{\infty} n (\mp)^n z^n Q_n/n}{z Q_1 + \underbrace{\sum_{1}^{\infty} n (\mp)^n z^{n+1} Q_{n+1}}_{\sum_{2}^{\infty} n (\mp)^{n-1} z^n Q_n}}$$

$m = n+1$

$$\frac{\beta P V}{\langle N\rangle} = \frac{z Q_1 \mp \sum_{2}^{\infty} n (\mp)^n z^n Q_n/n}{z Q_1 \mp \sum_{2}^{\infty} n (\mp)^n z^n Q_n}$$

$$\frac{\beta P V}{\langle N\rangle} = \frac{1 \mp \sum_{2}^{\infty} n (\mp)^n z^{n-1} Q_n/Q_1 n}{1 \mp \sum_{2}^{\infty} n (\mp)^n z^{n-1} Q_n/Q_1}$$

Expand to first non trivial order

$$\frac{\beta P V}{\langle N\rangle} = \frac{1 \mp z \frac{Q_2}{2 Q_1}}{1 \mp z \frac{Q_2}{Q_1}} \simeq \left(1 \mp z \frac{Q_2}{2Q_1}\right)\left(1 \pm z \frac{Q_2}{Q_1}\right)$$

$$\frac{\beta P V}{\langle N\rangle} = 1 \mp z \frac{Q_2}{2 Q_1} \pm z \frac{Q_2}{Q_1} + \mathcal{O}(z^3)$$

$$\frac{\beta P V}{\langle N\rangle} = 1 \pm \frac{1}{2} \frac{Q_2}{Q_1} z + \mathcal{O}(z^3)$$

Now to first order $\langle N\rangle = z Q_1$

$$\frac{\beta PV}{\langle N\rangle} = 1 \pm \frac{1}{2}\frac{Q_2}{Q_1}\frac{\langle N\rangle}{Q_1}$$

$$\boxed{\frac{\beta PV}{\langle N\rangle} = 1 \pm \frac{1}{2}\frac{Q_2}{Q_1^2}\langle N\rangle}$$

$$Q_2 = cV\int d\epsilon\,\epsilon^\alpha e^{-2\beta\epsilon} = \frac{cV}{\beta^{\alpha+1}}\int_0^\infty dx\, x^\alpha e^{-2x}$$

$$Q_1 = cV\int d\epsilon\,\epsilon^\alpha e^{-\beta\epsilon} = \frac{cV}{\beta^{\alpha+1}}\int_0^\infty dx\, x^\alpha e^{-x}$$

$$\frac{Q_2}{Q_1^2} = \frac{cV}{\beta^{\alpha+1}}\left(\frac{\beta^{\alpha+1}}{cV}\right)^2\frac{\int dx\, x^\alpha e^{-2x}}{\int_0^\infty dx\, x^\alpha e^{-x}}$$

$$\frac{\beta PV}{\langle N\rangle} = 1 \pm \frac{1}{2}\frac{\langle N\rangle}{V}\frac{\beta^{\alpha+1}}{c}\frac{1}{2^{\alpha+1}}$$

for non relativistic massive particles

$$Q_1 = \frac{V}{\lambda^3(T)} \qquad Q_2 = \frac{V}{\lambda^3\left(\frac{T}{2}\right)}$$

$$\frac{\beta PV}{\langle N\rangle} = 1 \pm \frac{1}{2}\frac{\langle N\rangle}{V}\frac{\lambda^6(T)}{\lambda^3(T/2)} = 1 \pm \frac{1}{2}\frac{\langle N\rangle}{V}\lambda^3(T)\left(\frac{\lambda(T)}{\lambda(T/2)}\right)^3$$

$$\boxed{\frac{\beta P V}{\langle N \rangle} = 1 \pm \frac{1}{2} \frac{\langle N \rangle}{V} \lambda^3(T) \frac{1}{2^{3/2}}}$$

$$\boxed{B_2(T) = \pm \frac{1}{2^{5/2}} \lambda^3(T)}$$

"repulsive" Fermi

"attractive" Bose