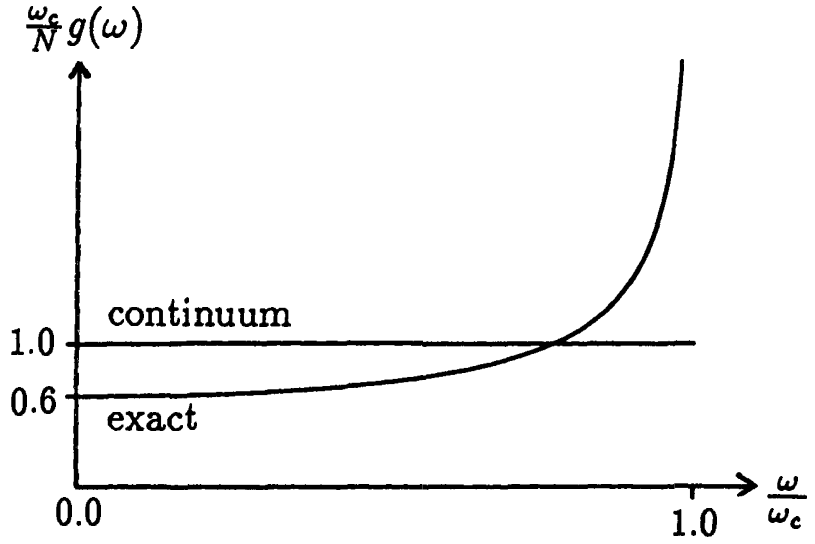

Fig. 7.23. Plots of the exact density of states and the density of states obtained using the continuum approximation, both for the one-dimensional lattice. Here we take $\omega_c = \omega_L = \omega_D$.

where in the second term we have again made the change of variables, $x = \beta\hbar\omega$, and have defined the Debye temperature, $T_D = \hbar\omega_D/k_B$.

In the limit $T \to 0$, the heat capacity in the continuum approximation becomes

$$C_N = \frac{Nk_BT}{T_D}\int_0^\infty \frac{dx\,x^2e^x}{(e^x-1)^2} + O(T^3) = \frac{Nk_BT\pi^2}{3T_D} + O(T^3). \tag{7.207}$$

At very low temperatures the continuum approximation also gives a heat capacity for the one-dimensional lattice which goes to zero linearly with the temperature. The coefficient differs slightly from the exact result in Eq. (7.201).

▶ S7.B. Momentum Condensation in an Interacting Fermi Fluid [15–17]

An ideal Bose–Einstein gas can condense in momentum space and thereby undergo a phase transition, but an ideal Fermi–Dirac gas is prevented from doing so because the Pauli exclusion principle does not allow more than one fermion to occupy a given quantum state. Electrons in a conducting solid are free to wander through the lattice and form a Fermi fluid. At low temperatures the electrons form a Fermi sea and only those near the Fermi surface affect the thermodynamic properties of the electron fluid (cf. Section 7.H). The electrons experience a mutual Coulomb repulsion which is screened by lattice ions. However, as first noted by Frohlich [18], those electrons in the neighborhood of the Fermi surface also experience a lattice-phonon-mediated effective attraction

(two electrons may in effect be attracted to one another because they are both attracted to the same lattice ion). Cooper [15] showed that this effective attraction at the Fermi surface could cause bound pairs of electrons to form, and these pairs could then condense in momentum space, giving rise to a phase transition in the interacting Fermi fluid. Bardeen, Schrieffer and Cooper, [16] showed that this momentum space condensation of *Cooper pairs* is the origin of superconductivity in materials. In 1972, they received the Nobel Prize for this work.

We shall now derive the thermodynamic properties of a Fermi fluid which can form Cooper pairs. It is found experimentally that Cooper pairs have zero total angular momentum and zero total spin. If the pairs are not undergoing a net translation through the fluid (no supercurrent), then we can assume that only those electrons at the Fermi surface with equal and opposite momentum and opposite spin components are attracted to one another. We shall assume that all other electrons behave like an ideal gas. With these assumptions, we can write the Hamiltonian of the electron fluid in the form

$$\hat{H} = \sum_{\mathbf{k},\lambda} \varepsilon_{\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k},\lambda} \hat{a}_{-\mathbf{k},\lambda} + \sum_{\mathbf{k}} \sum_{\mathbf{l}} V_{\mathbf{k},\mathbf{l}} \hat{a}^{\dagger}_{\mathbf{k},\uparrow} \hat{a}^{\dagger}_{-\mathbf{k},\downarrow} \hat{a}_{-\mathbf{l},\downarrow} \hat{a}_{\mathbf{l},\uparrow}, \tag{7.208}$$

where $\varepsilon_{\mathbf{k}} = \hbar^2 k^2/2m$, and λ denotes the z component of spin of a given electron and takes values $\lambda = \uparrow$ or $\lambda = \downarrow$ (spin component $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$, respectively). The operators, $\hat{a}^{\dagger}_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}$, respectively create and annihilate an electron with momentum $\hbar\mathbf{k}$ and spin component λ (cf. Appendix B). They satisfy fermion anticommutation relations. The interaction term in Eq. (7.208) destroys a pair of electrons with momenta $\hbar\mathbf{l}$ and $-\hbar\mathbf{l}$ and opposite spin components, and it creates a pair of electrons with momenta $\hbar\mathbf{k}$ and $-\hbar\mathbf{k}$ and opposite spin components. Since the electrons only experience an attraction at the Fermi surface, the interaction energy, $V_{\mathbf{k},\mathbf{l}}$, can be written

$$V_{\mathbf{k},\mathbf{l}} = \begin{cases} -V_0 & \text{if } |\mu' - \varepsilon_{\mathbf{k}}| \leqslant \Delta\varepsilon \text{ and } |\mu' - \varepsilon_{\mathbf{l}}| \leqslant \Delta\varepsilon, \\ 0 & \text{otherwise,} \end{cases} \tag{7.209}$$

where V_0 is a positive constant, μ' is the chemical potential of the fermi fluid, and $\Delta\varepsilon$ is a small energy of order $k_B T$.

In order to simplify our calculations, we shall compute the thermodynamic properties of this interacting Fermi fluid in the mean field approximation. We write the Hamiltonian in the form

$$\hat{H}_{mf} = \sum_{\mathbf{k},\lambda} \varepsilon_{\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda} + \sum_{\mathbf{k}} \Delta^{*}_{\mathbf{k}} \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow} + \sum_{\mathbf{k}} \Delta_{\mathbf{k}} \hat{a}^{\dagger}_{\mathbf{k},\uparrow} \hat{a}^{\dagger}_{-\mathbf{k},\downarrow}, \tag{7.210}$$

where

$$\Delta_{\mathbf{k}} = \begin{cases} \Delta & \text{if } |\mu - \varepsilon_{\mathbf{k}}| \leqslant \Delta\varepsilon, \\ 0 & \text{otherwise} \end{cases} \tag{7.211}$$

and

$$\Delta \equiv -V_0 \sum_{\mathbf{k}}{}' \langle \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow} \rangle \quad \text{and} \quad \Delta^* \equiv -V_0 \sum_{\mathbf{k}}{}' \langle \hat{a}^\dagger_{\mathbf{k},\uparrow} \hat{a}^\dagger_{-\mathbf{k},\downarrow} \rangle. \tag{7.212}$$

The prime on the summation, $\sum'_{\mathbf{k}}$, means that the summation is restricted to a distance, $\Delta\varepsilon$, on either side of the Fermi surface. The average, $\langle \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow} \rangle$, is defined as

$$\langle \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow} \rangle = \mathrm{Tr}[\hat{\rho} \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow}], \tag{7.213}$$

where the density operator, $\hat{\rho}$, is defined as

$$\hat{\rho} = \frac{e^{-\beta(\hat{H}_{mf} - \mu\hat{N})}}{\mathrm{Tr}[e^{-\beta(\hat{H}_{mf} - \mu\hat{N})}]}. \tag{7.214}$$

The average, $\langle \hat{a}^\dagger_{\mathbf{k},\uparrow} \hat{a}^\dagger_{-\mathbf{k},\downarrow} \rangle$, is similarly defined. The number operator, $\hat{N}$, is defined as

$$\hat{N} = \sum_{\mathbf{k},\lambda} \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}. \tag{7.215}$$

The quantity Δ is called the *gap function* and may be real or complex. It is a thermodynamic quantity and is a measure of the average binding energy of all the Cooper pairs. If a macroscopic number of Cooper pairs form, then $\langle \hat{a}^\dagger_{\mathbf{k},\uparrow} \hat{a}^\dagger_{-\mathbf{k},\downarrow} \rangle \approx \sqrt{n_c}$ and $\langle \hat{a}^\dagger_{\mathbf{k},\uparrow} \hat{a}^\dagger_{-\mathbf{k},\downarrow} \rangle \approx \sqrt{n_c}$ where n_c is the average number of Cooper pairs in the fluid. Δ is the order parameter for this transition.

It is important to notice that the Hamiltonian, $\hat{H}_{mf}$, does not commute with the number operator, $\hat{N}$, if $\Delta \neq 0$. This means that if a macroscopic number of Cooper pairs form, the system does not conserve the particle (electron) number and the gauge symmetry is broken. The formation of a macroscopic number of Cooper pairs is a phase transition somewhat analogous to Bose–Einstein condensation (cf. Section 7.H). In both cases, gauge symmetry is broken. Since we are working in the grand canonical ensemble and only specify the average particle number, the fact that gauge symmetry is broken is not a problem. If a macroscopic number of Cooper pairs form, the total energy of the system is lowered. The transition to the condensed phase occurs when the thermal energy, $k_B T$, which tends to break Cooper pairs apart, becomes less important than the phonon-mediated attraction between electrons.

It is useful now to introduce an effective Hamiltonian

$$\hat{K} = \sum_{\mathbf{k}} \xi_{\mathbf{k}} (\hat{a}^\dagger_{\mathbf{k},\uparrow} \hat{a}_{\mathbf{k},\uparrow} - \hat{a}_{-\mathbf{k},\downarrow} \hat{a}^\dagger_{-\mathbf{k},\downarrow}) + \sum_{\mathbf{k}} \Delta^*_{\mathbf{k}} \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow} + \sum_{\mathbf{k}} \Delta_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k},\uparrow} \hat{a}^\dagger_{-\mathbf{k},\downarrow}, \tag{7.216}$$

where

$$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu' = \frac{\hbar^2 k^2}{2m} - \mu', \tag{7.217}$$

and we have made use of the fermion anticommutation relations. The effective Hamiltonian, $\hat{K}$, differs from $\hat{H}_{mf} - \mu\hat{N}$ only by a constant term. Therefore the density operator can also be written

$$\hat{\rho} = \frac{e^{-\beta(\hat{H}_{mf}-\mu\hat{N})}}{\mathrm{Tr}\left[e^{-\beta(\hat{H}_{mf}-\mu\hat{N})}\right]} = \frac{e^{\beta\hat{K}}}{\mathrm{Tr}\left[e^{-\beta\hat{K}}\right]}. \tag{7.218}$$

The effective Hamiltonian, $\hat{K}$, can be written in matrix form:

$$\hat{K} = \sum_{\mathbf{k}} \bar{\alpha}_{\mathbf{k}}^{\dagger} \bar{K}_{\mathbf{k}} \bar{\alpha}_{\mathbf{k}}, \tag{7.219}$$

where

$$\bar{K}_{\mathbf{k}} \equiv \begin{pmatrix} \xi_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & -\xi_{\mathbf{k}} \end{pmatrix}, \qquad \bar{\alpha}_{\mathbf{k}} = \begin{pmatrix} \hat{a}_{\mathbf{k},\uparrow} \\ \hat{a}_{-\mathbf{k},\downarrow}^{\dagger} \end{pmatrix}, \qquad \bar{\alpha}_{\mathbf{k}}^{\dagger} = (\hat{a}_{\mathbf{k},\uparrow}^{\dagger} \quad \hat{a}_{-\mathbf{k},\downarrow}). \tag{7.220}$$

As was first shown by Bogoliubov [19], the effective Hamiltonian, $\hat{K}$, can be diagonalized by means of a unitary transformation which preserves the fermion anticommutation relations. In so doing, we obtain the Hamiltonian for effective excitations (called bogolons) of the system. To diagonalize the effective Hamiltonian, we introduce a 2×2 unitary matrix,

$$\bar{U}_{\mathbf{k}} \equiv \begin{pmatrix} u_{\mathbf{k}}^* & v_{\mathbf{k}} \\ -v_{\mathbf{k}}^* & u_{\mathbf{k}} \end{pmatrix}, \tag{7.221}$$

Since $\bar{U}_{\mathbf{k}}^{\dagger}\bar{U}_{\mathbf{k}} = \bar{U}_{\mathbf{k}}\bar{U}_{\mathbf{k}}^{\dagger} = \bar{1}$ (unitarity), we must have $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$. We also introduce the vectors

$$\bar{\Gamma}_{\mathbf{k}} = \begin{pmatrix} \hat{\gamma}_{\mathbf{k},0} \\ \hat{\gamma}_{\mathbf{k},1}^{\dagger} \end{pmatrix}, \qquad \bar{\Gamma}_{\mathbf{k}}^{\dagger} = (\hat{\gamma}_{\mathbf{k},0}^{\dagger} \ \hat{\gamma}_{\mathbf{k},1}), \tag{7.222}$$

which are related to the vectors, $\bar{\alpha}_{\mathbf{k}}$, via the unitary transformation

$$\bar{\alpha}_{\mathbf{k}} = \bar{U}_{\mathbf{k}}\bar{\Gamma}_{\mathbf{k}}. \tag{7.223}$$

The physical significance of the vectors, $\bar{\Gamma}_{\mathbf{k}}$, will become clear below. It is easy to show that since $\hat{a}_{\mathbf{k},\lambda}^{\dagger}$ and $\hat{a}_{\mathbf{k},\lambda}$ obey fermion anticommutation relations, the

operators, $\hat{\gamma}^{\dagger}_{\mathbf{k},\mathbf{i}}$ and $\hat{\gamma}_{\mathbf{k},\mathbf{i}}$ $(i = 0, 1)$, must also obey fermion anticommutation relations

$$[\hat{\gamma}_{\mathbf{k},\mathbf{i}}, \hat{\gamma}^{\dagger}_{\mathbf{k}',\mathbf{i}'}]_+ = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{i},\mathbf{i}'}, \qquad [\hat{\gamma}_{\mathbf{k},\mathbf{i}}, \hat{\gamma}_{\mathbf{k}',\mathbf{i}'}]_+ = [\hat{\gamma}^{\dagger}_{\mathbf{k},\mathbf{i}}, \hat{\gamma}^{\dagger}_{\mathbf{k}',\mathbf{i}'}]_+ = 0. \tag{7.224}$$

If we revert Eq. (7.223), we see that $\hat{\gamma}_{\mathbf{k},0}$ decreases the momentum of the system by $\hbar\mathbf{k}$ and lowers the spin by $\hbar$ (it destroys a particle with quantum numbers, $(\mathbf{k}, \uparrow)$, and creates one with quantum numbers, $(-\mathbf{k}, \downarrow)$, whereas $\hat{\gamma}_{\mathbf{k},1}$ increases the momentum of the system by $\hbar\mathbf{k}$ and raises the spin by $\hbar$.

We now require that the unitary matrix, $\bar{U}_{\mathbf{k}}$, diagonalize the effective Hamiltonian, $\bar{K}_K$. That is,

$$\bar{U}^{\dagger}_{\mathbf{k}}\bar{K}_{\mathbf{k}}\bar{U}_{\mathbf{k}} = \bar{E}_{\mathbf{k}} \qquad \text{with } \bar{E}_{\mathbf{k}} = \begin{pmatrix} E_{\mathbf{k},0} & 0 \\ 0 & E_{\mathbf{k},1} \end{pmatrix}. \tag{7.225}$$

We find that $E_{\mathbf{k},0} = E_{\mathbf{k}}$ and $E_{\mathbf{k},1} = -E_{\mathbf{k}}$ with

$$E_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2}. \tag{7.226}$$

With this transformation, we have succeeded in reducing the interacting Fermi gas of electrons to an ideal Fermi gas of bogolons. In terms of bogolon operators, the effective Hamiltonian becomes

$$\begin{aligned} \hat{K} &= \sum_{\mathbf{k}} \bar{\alpha}^{\dagger}_{\mathbf{k}} \bar{U}_{\mathbf{k}} \bar{U}^{\dagger}_{\mathbf{k}} \bar{K}_{\mathbf{k}} \bar{U}_{\mathbf{k}} \bar{U}^{\dagger}_{\mathbf{k}} \bar{\alpha}_{\mathbf{k}} = \sum_{\mathbf{k}} \bar{\gamma}^{\dagger}_{\mathbf{k}} \bar{E}_{\mathbf{k}} \bar{\gamma}_{\mathbf{k}} \\ &= \sum_{\mathbf{k}} (E_{\mathbf{k},0}\, \hat{\gamma}^{\dagger}_{\mathbf{k},0}\, \hat{\gamma}_{\mathbf{k},0} - E_{\mathbf{k},1}\, \hat{\gamma}^{\dagger}_{\mathbf{k},1}\, \hat{\gamma}_{\mathbf{k},1} + E_{\mathbf{k},1}). \end{aligned} \tag{7.227}$$

The effective Hamiltonian, when written in terms of bogolon operators, looks like that of an ideal Fermi gas. The bogolons are collective modes of the system and play a role analogous to that of phonons in a Debye solid, although their dispersion relation is quite different.

We can now obtain a self-consistent equation for the gap function, Δ. First note that

$$\langle \hat{\gamma}^{\dagger}_{\mathbf{k},0} \hat{\gamma}_{\mathbf{k},0} \rangle = \frac{1}{(1 + e^{\beta E_{\mathbf{k},0}})} = \frac{1}{2}\left[1 - \tanh\left(\frac{\beta E_{\mathbf{k},0}}{2}\right)\right] \tag{7.228}$$

and

$$\langle \hat{\gamma}^{\dagger}_{\mathbf{k},1} \hat{\gamma}_{\mathbf{k},1} \rangle = \frac{1}{(1 + e^{-\beta E_{\mathbf{k},1}})} = \frac{1}{2}\left[1 + \tanh\left(\frac{\beta E_{\mathbf{k},1}}{2}\right)\right] \tag{7.229}$$

Then

$$\langle \bar{\alpha}_{\mathbf{k}} \bar{\alpha}_{\mathbf{k}}^{\dagger} \rangle = \begin{pmatrix} 1 - \langle \hat{a}_{\mathbf{k},\uparrow}^{\dagger} \hat{a}_{\mathbf{k},\uparrow} \rangle & -\langle \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow} \rangle \\ -\langle \hat{a}_{\mathbf{k},\uparrow}^{\dagger} \hat{a}_{-\mathbf{k},\downarrow}^{\dagger} \rangle & \langle \hat{a}_{-\mathbf{k},\downarrow}^{\dagger} \hat{a}_{-\mathbf{k},\downarrow} \rangle \end{pmatrix}$$

$$= \bar{U}_{\mathbf{k}} \begin{pmatrix} 1 - \langle \hat{\gamma}_{\mathbf{k},0}^{\dagger} \hat{\gamma}_{\mathbf{k},0} \rangle & 0 \\ 0 & \langle \hat{\gamma}_{\mathbf{k},1}^{\dagger} \hat{\gamma}_{\mathbf{k},1} \rangle \end{pmatrix} \bar{U}_{\mathbf{k}}^{\dagger}$$

$$= \frac{1}{2} \hat{1} + \frac{1}{2} \bar{U}_{\mathbf{k}} \begin{pmatrix} \tanh\,(\beta E_{\mathbf{k}}/2) & 0 \\ 0 & -\tanh\,(\beta E_{\mathbf{k}}/2) \end{pmatrix} \bar{U}_{\mathbf{k}}^{\dagger}$$

$$= \frac{1}{2} \hat{1} + \frac{1}{2E_{\mathbf{k}}} \bar{U}_{\mathbf{k}} \begin{pmatrix} E_{\mathbf{k}} & 0 \\ 0 & -E_{\mathbf{k}} \end{pmatrix} \bar{U}_{\mathbf{k}}^{\dagger} \tanh\,(\beta E_{\mathbf{k}}/2)$$

$$= \frac{1}{2} \hat{1} + \frac{1}{2E_{\mathbf{k}}} \bar{K}_{\mathbf{k}} \tanh\,(\beta E_{\mathbf{k}}/2). \tag{7.230}$$

Let us now equate off-diagonal matrix elements. We write

$$-\langle \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow} \rangle = \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} \tanh\,(\beta E_{\mathbf{k}}/2) \tag{7.231}$$

If we multiply Eq. (7.231) by $V_{\mathbf{k},\mathbf{l}}$, integrate over $\mathbf{k}$, and use Eqs. (7.211) and (7.212), we obtain

$$1 = V_0 \sum_{\mathbf{k}}{}' \frac{1}{2E_{\mathbf{k}}} \tanh\,(\beta E_{\mathbf{k}}/2). \tag{7.232}$$

It is useful to note that under the primed summation the bogolon energy can be written $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta|^2}$. Equation (7.232) is the equation for the gap function. It has been obtained from the grand canonical ensemble. Therefore, the solutions of the gap equation correspond to extrema of the free energy. The solution at a given temperature which corresponds to the stable thermodynamic state is the one which minimizes the free energy. Since the energy, $E_{\mathbf{k}}$, depends on the gap, Eq. (7.232) is rather complicated.

Let us now determine some properties of the gap function from Eq. (7.232). If we assume that the system is contained in a large volume, V, we can change the summation to an integration [cf. Eq. (7.162)]. Note that

$$\sum_{\mathbf{k}} \approx \frac{V}{2\pi^2} \int_0^{\infty} dk\, k^2 = \frac{m^{3/2} V}{\sqrt{2}\pi^2 \hbar^3} \int_{-\mu'}^{\infty} d\xi_{\mathbf{k}} \sqrt{\xi_{\mathbf{k}} + \mu'}, \tag{7.233}$$

where we have Eg. (7.217). The summation, $\sum_{\mathbf{k}}'$, which is restricted to the

neighborhood of the Fermi surface, can be written

$$\sum_{\mathbf{k}}{}' \approx \frac{m^{3/2}V}{\sqrt{2}\pi^2\hbar^3}\int_{-\Delta\varepsilon}^{\Delta\varepsilon}\sqrt{\varepsilon_f}\,d\xi_{\mathbf{k}} \approx N(0)\int_{-\Delta\varepsilon}^{\Delta\varepsilon} d\xi_{\mathbf{k}}, \tag{7.234}$$

where we have set $\mu \approx \varepsilon_f$ (ε_f is the Fermi energy) and $N(0) = mVk_f/2\pi^2\hbar^2$ is the density of states at the Fermi surface for a single spin state (cf. Exercise 7.9). We can now write Eq. (7.232) as

$$1 = V_0 N(0)\int_0^{\Delta\varepsilon} d\xi_{\mathbf{k}} \frac{\tanh\left[\frac{\beta}{2}\sqrt{\xi_{\mathbf{k}}^2 + |\Delta(T)|^2}\right]}{\sqrt{\xi_{\mathbf{k}}^2 + |\Delta(T)|^2}}. \tag{7.235}$$

Equation (7.235) determines the temperature dependence of the gap, $\Delta(T)$, and can be used to find the transition temperature.

The energy of bogolons (measured from the Fermi surface) with momentum $\hbar\mathbf{k}$ is $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta(T)|^2}$. It takes a finite energy to excite them, regardless of their momentum, because there is a gap in the energy spectrum. At the critical temperature, T_c, the gap goes to zero and the excitation spectrum reduces to that of an ideal Fermi gas. The critical temperature can be obtained from Eq. (7.235). It is the temperature at which the gap becomes zero. Thus, at the critical temperature we have

$$\begin{aligned} 1 &= V_0 N(0)\int_0^{\Delta\varepsilon} d\xi_{\mathbf{k}} \frac{\tanh\left[\frac{\beta\xi_{\mathbf{k}}}{2}\right]}{\xi_{\mathbf{k}}} = N(0)V_0\int_0^{\beta_c\Delta\varepsilon/2} dx \frac{\tanh(x)}{x} \\ &= N(0)V_0 \ln\left[\frac{\alpha}{2}\beta_c\Delta\varepsilon\right], \end{aligned} \tag{7.236}$$

where $\beta_c = (k_B T_c)^{-1}$, $\alpha = 2.26773$, and we have used the fact that

$$\int_0^b \frac{\tanh(x)}{x} dx = \ln(\alpha b), \tag{7.237}$$

for $b > 100$. Thus, Eq. (7.236) holds when $\beta_c\Delta\varepsilon/2 > 100$. This means that $N(0)V_0 < 0.184$ and therefore use of Eq. (7.236) restricts us to fairly weakly coupled systems. From Eqs. (7.236) and (7.237) we obtain

$$k_B T_c = \frac{\alpha}{2}\Delta\varepsilon e^{-1/N(0)V_0}, \tag{7.238}$$

for $\beta_c\Delta\varepsilon/2 > 100$. Thus, the critical temperature, T_c, varies exponentially with the strength of the attractive interaction.

We can also use Eq. (7.235) to find the gap, $\Delta(0) \equiv \Delta_0$, at $T = 0$ K. Since $\tanh(\infty) = 1$, we can write

$$1 = V_0 N(0) \int_0^{\Delta\varepsilon} d\xi_{\mathbf{k}} \frac{1}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta_0^2}} = V_0 N(0) \sinh^{-1}\left(\frac{\Delta\varepsilon}{\Delta_0}\right), \tag{7.239}$$

or

$$\Delta_0 = \frac{\Delta\varepsilon}{\sinh\,(1/V_0 N(0))} \approx 2\Delta\varepsilon e^{-1/N(0)V_0}. \tag{7.240}$$

The rightmost expression for Δ_0 applies for weakly coupled systems when $N(0)V_0 < 0.184$. Comparing Eqs. (7.238) and (7.240), we obtain the following relation between the critical temperature and the zero temperature gap *for weakly coupled systems*:

$$\frac{\Delta_0}{k_B T_c} = \frac{4}{\alpha} = 1.764. \tag{7.241}$$

Equation (7.241) is in good agreement with experimental values of this ratio for superconductors. Equation (7.235) may be solved numerically to obtain a plot of the gap as a function of temperature. The gap function is a real function for the case (such as we are considering here) when no supercurrent is present. We show the behavior of $\Delta(T)$ in Fig. 7.24 for weakly coupled systems.

Since bogolons form an ideal gas, the entropy can be written in the form

$$S = -2k_B \sum_{\mathbf{k}} [n_{\mathbf{k}} \ln(n_{\mathbf{k}}) + (1 - n_{\mathbf{k}}) \ln(1 - n_{\mathbf{k}})], \tag{7.242}$$

where $n_{\mathbf{k}} = (1 + e^{\beta E_{\mathbf{k}}})^{-1}$ (cf. Problem 7.23). The heat capacity, $C_{V,N}$, is easy to

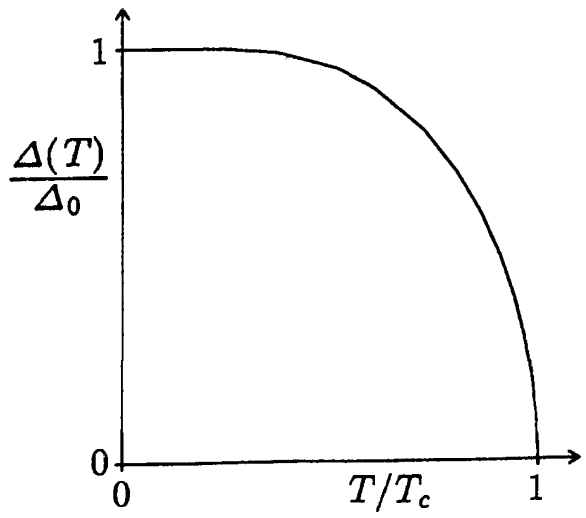

Fig. 7.24. A plot of the ratio $\Delta(T)/\Delta_0$ versus the reduced temperature, T/T_c, for a weakly coupled system.

find from Eq. (7.242). let us first note that for a Fermi gas at very low temperature we have $\mu \approx \varepsilon_f$, where ε_f is the Fermi energy, and $(\partial\mu/\partial T)_{V,\langle N\rangle} \approx 0$. Thus,

$$\begin{aligned} C_{V,N} &= T\left(\frac{\partial S}{\partial T}\right)_{V,\langle N\rangle} \approx 2\beta k_B \sum_{\mathbf{k}} \frac{\partial n_{\mathbf{k}}}{\partial \beta} \ln\left(\frac{n_{\mathbf{k}}}{1-n_{\mathbf{k}}}\right) \\ &= -2\beta k_B \sum_{\mathbf{k}} \frac{\partial n_{\mathbf{k}}}{\partial E_{\mathbf{k}}}\left(E_{\mathbf{k}}^2 + \frac{1}{2}\beta \frac{\partial |\Delta_{\mathbf{k}}|^2}{\partial \beta}\right). \end{aligned} \tag{7.243}$$

We can now examine the heat capacity, both at the critical temperature and in the limit $T \to 0$ K.

Let us first look at the neighborhood of the critical point. The first term in Eq. (7.243) is continuous at $T = T_c$, but the second term is not since $\partial|\Delta_{\mathbf{k}}|^2/\partial\beta$ has a finite value for $T < T_c$ but is zero for $T > T_c$. Near $T = T_c$, we may let $E_{\mathbf{k}} \to |\xi_{\mathbf{k}}|$. Then the heat capacity just below the critical temperature is

$$C_{V,N}^{<} \approx -2\beta_c k_B \sum_{\mathbf{k}} \frac{\partial n_{\mathbf{k}}}{\partial |\xi_{\mathbf{k}}|}\left(\xi_{\mathbf{k}}^2 + \frac{1}{2}\beta_c \left(\frac{\partial |\Delta_{\mathbf{k}}|^2}{\partial \beta}\right)_{T=T_c}\right), \tag{7.244}$$

and just above the critical temperature it is

$$C_{V,N}^{>} \approx -2\beta_c k_B \sum_{\mathbf{k}} \frac{\partial n_{\mathbf{k}}}{\partial |\xi_{\mathbf{k}}|} \xi_{\mathbf{k}}^2. \tag{7.245}$$

The discontinuity in the heat capacity at the critical temperature is

$$\begin{aligned} \Delta C_{V,N} &= C_{V,\langle N\rangle}^{<} - C_{V,\langle N\rangle}^{>} = -\beta_c^2 k_B {\sum_{\mathbf{k}}}' \left(\frac{\partial |\Delta|^2}{\partial \beta}\right)_{T=T_c} \frac{\partial n_{\mathbf{k}}(|\xi_{\mathbf{k}}|)}{\partial |\xi_{\mathbf{k}}|} \\ &= -k_B \beta_c^2 N(0)\left(\frac{\partial |\Delta|^2}{\partial \beta}\right)_{T=T_c} = N(0)\left(\frac{\partial |\Delta|^2}{\partial T}\right)_{T=T_c} \end{aligned} \tag{7.246}$$

Thus, the heat capacity has a finite discontinuity at the critical temperature, as we would expect for a mean field theory.

Let us now compute the heat capacity in the limit $T \to 0$. As we can see from Fig. 7.24, the gap function, Δ, approaches a finite value, Δ_0, as $T \to 0$ and $\partial\Delta/\partial T \to 0$ as $T \to 0$. As a result the heat capacity takes a fairly simple form in the limit $T \to 0$. If we assume that $\mu' \approx \varepsilon_f$ and $\Delta \approx \Delta_0$ in Eq. (7.243), then the heat capacity takes the form

$$C_{V,N} \approx -2\beta k_B \sum_{\mathbf{k}} E_{\mathbf{k}}^2 \frac{\partial n_{\mathbf{k}}}{\partial E_{\mathbf{k}}} = 2\beta^2 k_B \sum_{\mathbf{k}} E_{\mathbf{k}}^2 \frac{e^{\beta E_{\mathbf{k}}}}{(1+e^{\beta E_{\mathbf{k}}})^2}, \tag{7.247}$$

where $E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta_0^2}$. In order to change Eq. (7.247) it is useful to introduce the bogolon density of states. We can write

$$\sum_{\mathbf{k}} = \frac{m^{3/2}V}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty \sqrt{\varepsilon_{\mathbf{k}}}d\varepsilon_{\mathbf{k}} = \frac{m^{3/2}V}{\sqrt{2}\pi^2\hbar^3}\int_{\Delta_0}^\infty \left[\sqrt{E_{\mathbf{k}}^2 - \Delta_0^2} + \varepsilon_f\right]^{1/2} \frac{E_{\mathbf{k}}dE_{\mathbf{k}}}{\sqrt{E_{\mathbf{k}}^2 - \Delta_0^2}}. \tag{7.248}$$

For momenta, $k \approx k_f$, the density of states is singular. Therefore, the dominant contribution to the integral comes from the neighborhood of the Fermi surface and we can write

$$\sum_{\mathbf{k}} \approx N(0)\int_{\Delta_0}^\infty \frac{E_{\mathbf{k}}dE_{\mathbf{k}}}{\sqrt{E_{\mathbf{k}}^2 - \Delta_0^2}}. \tag{7.249}$$

Let us next note that in the limit $T \to 0$ we can write $e^{\beta E_{\mathbf{k}}}/(1 + e^{\beta E_{\mathbf{k}}})^2 \approx e^{-\beta E_{\mathbf{k}}}$. Thus, the heat capacity takes the form

$$C_{V,N} \approx 2\beta^2 k_B N(0)\int_{\Delta_0}^\infty \frac{E_{\mathbf{k}}^3 dE_{\mathbf{k}}}{\sqrt{E_{\mathbf{k}}^2 - \Delta_0^2}} e^{-\beta E_{\mathbf{k}}}. \tag{7.250}$$

The integral in Eq. (7.250) is easy to do if we use a trick. Note that

$$I \equiv \int_{\Delta_0}^\infty \frac{E_{\mathbf{k}}dE_{\mathbf{k}}}{\sqrt{E_{\mathbf{k}}^2 - \Delta_0^2}} e^{-\beta E_{\mathbf{k}}} = \Delta_0 K_1(\beta\Delta_0), \tag{7.251}$$

where K_1 is a modified Bessel function [19]. But

$$\int_{\Delta_0}^\infty \frac{E_{\mathbf{k}}^3 dE_{\mathbf{k}}}{\sqrt{E_{\mathbf{k}}^2 - \Delta_0^2}} e^{-\beta E_{\mathbf{k}}} = \frac{\partial^2 I}{\partial\beta^2} = \frac{1}{4}\Delta_0^3[3K_1(\beta\Delta_0) + K_3(\beta\Delta_0)]. \tag{7.252}$$

Thus, the heat capacity takes the form

$$C_{V,N} = \frac{1}{2}\beta^2 k_B N(0)\Delta_0^3[3K_1(\beta\Delta_0) + K_3(\beta\Delta_0)]. \tag{7.253}$$

If we now make use of the asymptotic form of the modified Bessel functions, $K_n(\beta\Delta_0) \approx \sqrt{\pi/2\beta\Delta_0}\, e^{-\beta\Delta_0}$, the heat capacity takes the form

$$C_{V,N} \approx \sqrt{2\pi}\beta^{3/2}k_B N(0)\Delta_0^{5/2}e^{-\beta\Delta_0} \tag{7.254}$$

in the limit $T \to 0$. Thus, the heat capacity of the condensed Fermi fluid goes to zero exponentially with temperature rather than linearly as in the case for an ideal Fermi gas. In Fig. 7.25 we show a sketch of the heat capacity of the interacting Fermi fluid (superconductor). The solid line is the Fermi fluid, and the dashed line is an ideal Fermi gas.

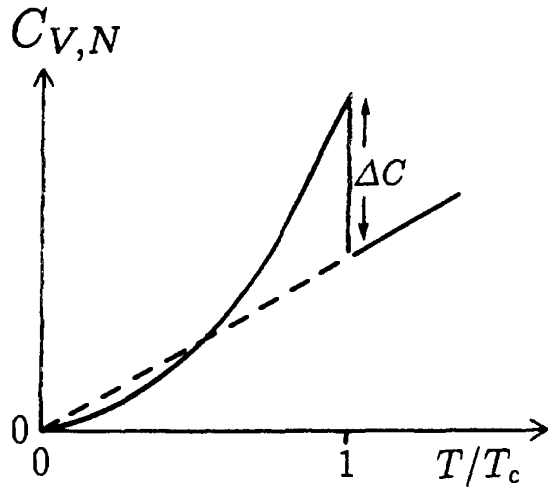

Fig. 7.25. A sketch of the heat capacity for a superconductor. The straight dashed line gives the heat capacity in the absence of interaction (ideal Fermi gas). The solid line shows the jump in the heat capacity at the critical point and the exponential decay for temperatures below the critical point.

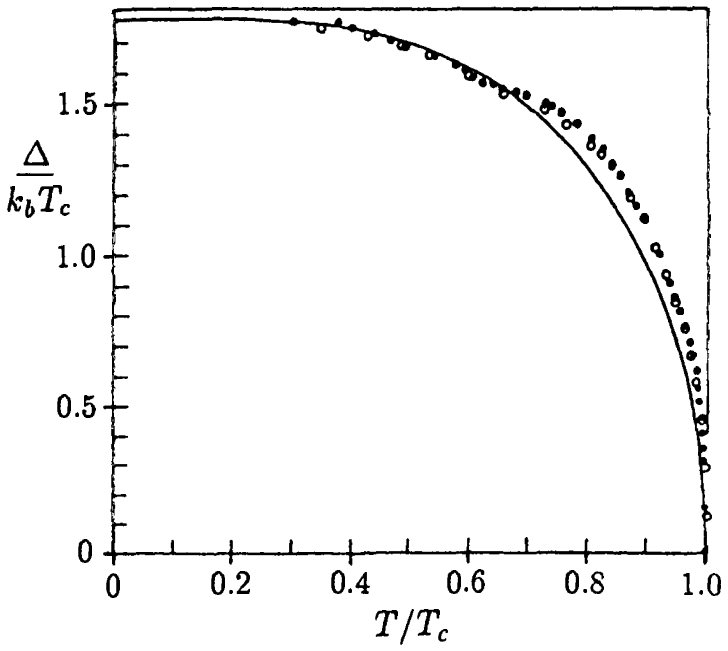

Fig. 7.26. Variation of $\Delta/k_B T_c$ with reduced temperature, T/T_c, for tin. The data points are obtained from ultrasonic acoustic attenuation measurements [20] for two different frequencies. The solid line is BCS theory. Reprinted, by permission, from R. W. Morse and H. V. Bohm, *Phys. Rev.* **108**, 1094 (1954).

The mean field theory gives a surprisingly good description of the behavior of real superconductors. In Fig. 7.26 we show experimental measurements of the gap function, Δ, as a function of temperature for tin. The solid line is the mean field theory of Bardeen, Cooper, and Schrieffer. The experimental points, which are obtained from ultrasonic accoustic attenuation measurements [21], fit it very well.