Step BCS

Linearize the potential

$$V_{BCS} = -V_0 \underbrace{\sum_k \theta_k c^+_{k\uparrow} c^+_{-k\downarrow}}_{A^+} \underbrace{\sum_{k'} \theta_{k'} c_{-k'\downarrow} c_{k'\uparrow}}_{A}$$

$$A = \underset{\text{number}}{\langle A \rangle} + \delta\hat{A} = X + \delta\hat{A}$$

$$A^+A = (X^* + \delta A)(X + \delta A) = |X|^2 + X^*\delta A + X\delta A^+ + \mathcal{O}(\delta A^2)$$

replace $\delta A = A - X$

$$A^+A \cong |X|^2 + X^*(A - X) + X(A^+ - X^*)$$

$$A^+A \simeq -|X|^2 + X^*A + XA^+$$

define

$$\Delta = \sum_k V_0 \theta_k \langle A_k \rangle = V_0 X \qquad A^+_k = c^+_{k\uparrow} c^+_{-k\downarrow} \quad A_k = c_{-k\downarrow} c_{k\uparrow}$$

$$\hat{V} \cong -V_0 A^+A = -V_0 \left[-\frac{|\Delta|^2}{V_0^2} + \frac{1}{V_0} \Delta^* A + \frac{1}{V_0} \Delta A^+ \right]$$

$$\hat{V}^{lin}_{BCS} = \frac{|\Delta|^2}{V_0} - \sum_k \left(\Delta^* \theta_k A_k + \Delta \theta_k A^+_k \right)$$

Separate low / high energies

$$\hat{K} = \hat{H} - \mu \hat{N} = \underbrace{\Sigma_k (\varepsilon_k - \mu)(c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow})} + V_{BCS}^{LIN}$$

use this only for low energy $\left\{ \boxed{\Sigma_k (\varepsilon_k - \mu) + \Sigma_k (\varepsilon_k - \mu)[c^\dagger_{k\uparrow} c_{k\uparrow} - c_{k\downarrow} c^\dagger_{k\downarrow}]} \right.$

$$\hat{K} = \Sigma_k (1-\Theta_k)(\varepsilon_k - \mu)(c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow}) + \frac{|\Delta|^2}{V_0} +$$

$$+ \Sigma_k \Theta_k (\varepsilon_k - \mu) + \Sigma_k \Theta_k (\varepsilon_k - \mu)[c^\dagger_{k\uparrow} c_{k\uparrow} - c_{k\downarrow} c^\dagger_{k\downarrow}]$$

$$- \Delta^* \Sigma_k \Theta_k \overbrace{c_{-k\downarrow} c_{k\uparrow}}^{A_k} + \Delta \Sigma_k \Theta_k \underbrace{c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}}_{A^\dagger_k}$$

$\{\gamma, \gamma^\dagger\} = 1$

$\gamma\gamma^\dagger = -\gamma^\dagger\gamma + 1$

Bogoliubov Diagonalisation

$$\Sigma_k \Theta_k E_k [\gamma^\dagger_{k0}\gamma_{k0} - \gamma_{k1}\gamma^\dagger_{k1}] =$$

$$= \underbrace{\Sigma_k \Theta_k E_k (\gamma^\dagger_{k0}\gamma_{k0} + \gamma^\dagger_{k1}\gamma_{k1})}_{\text{fermions with } \mu = 0} - \Sigma_k \Theta_k E_k$$

constant

$$\hat{K} = \cancel{\Sigma_k} \frac{|\Delta|^2}{V_0} + \Sigma_k \Theta_k (\varepsilon_k - \mu) - \Sigma_k \Theta_k E_k +$$

$$+ \Sigma_k (1-\Theta_k)(\varepsilon_k - \mu)[c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow}]$$

normal electrons outside

$$+ \Sigma_k \Theta_k E_k (\gamma^\dagger_{k0}\gamma_{k0} + \gamma^\dagger_{k1}\gamma_{k1})$$

$\mu = 0$ Bogolons inside

$$E_k = \sqrt{(\varepsilon_k - \mu)^2 + |\Delta|^2}$$

trace over fermions / Bogolons

$$\Psi(|\Delta|) = \frac{|\Delta|^2}{V_0} + \Sigma_k \Theta_k(\varepsilon_k-\mu) - \Sigma_k \Theta_k E_k$$

$$- \overset{\text{spin}}{\frac{2}{\beta}} \Sigma_k (1-\Theta_k) \ln\left(1+e^{-\beta(\varepsilon_k-\mu)}\right)$$

$$- \overset{\text{flavors}}{\frac{2}{\beta}} \Sigma_k \Theta_k \ln\left(1+e^{-\beta E_k}\right)$$

terms which do not depend on order parameter Δ

$$\Psi_0(T) = \Sigma_k \Theta_k(\varepsilon_k-\mu) - \frac{1}{\beta}\Sigma_k (1-\Theta_k)\ln\left(1+e^{-\beta(\varepsilon_k-\mu)}\right)$$

$$\Psi(|\Delta|) = \frac{|\Delta|^2}{V_0} - \Sigma_k \Theta_k E_k - \frac{1}{\beta}\Sigma_k \Theta_k \ln\left(1+e^{-\beta E_k}\right) + \Psi_0(T)$$

define

$E(\varepsilon) = \sqrt{\epsilon^2+|\Delta|^2}$ where $\boxed{\varepsilon \leftarrow \varepsilon_k-\mu}$

$N(\varepsilon) = \Sigma_k \delta(\varepsilon-\varepsilon_k+\mu)$ DoS around Fermi energy

$$\Sigma_k \Theta_k f(\varepsilon_k-\mu) = \int_{-\epsilon_d}^{\epsilon_d} d\epsilon\, N(\varepsilon)\, f(\varepsilon)$$

Expression in therms of DOS

define $E(\varepsilon) = \sqrt{\epsilon^2 + |\Delta|^2}$

use

$$E + \frac{2}{\beta}\ln\left(1 + e^{-\beta E}\right) = \frac{2}{\beta}\ln e^{\frac{\beta E}{2}} + \frac{2}{\beta}\ln\left(1 + e^{-\beta E}\right)$$

$$= \frac{2}{\beta}\ln 2\,\mathrm{ch}\left(\frac{\beta E}{2}\right)$$

$$\boxed{\Psi(\Delta) = \Psi_0(T) + \frac{|\Delta|^2}{V_0} - \frac{2}{\beta}\int_{-\epsilon_d}^{\epsilon_d} d\epsilon\, \mathcal{N}(\varepsilon)\ln 2\,\mathrm{ch}\left(\frac{\beta E(\varepsilon)}{2}\right)}$$

Ψ independent on the phase but depends on 2 components

$\Delta = V_0 \sum_k \theta_k \langle c_{-k\downarrow} c_{k\uparrow} \rangle$ comlex order parameter

$\Delta = 0$ normal phase
$\Delta \neq 0$ SC phase

$$\langle\ \rangle_F = \int_{\epsilon_d}^{\epsilon_d} d\epsilon\, \mathcal{N}(\varepsilon)(\ldots)$$

$$\Psi(\Delta) = \Psi_0(T) + \frac{|\Delta|^2}{V_0} - \frac{2}{\beta}\left\langle \ln 2\,\mathrm{ch}\left(\frac{\beta E(\varepsilon)}{2}\right)\right\rangle_F$$

Landau expansion of Ψ

small Δ expansion

$\ln 2\,ch\left(\frac{\beta E}{2}\right)$

$$E = \sqrt{\epsilon^2+\Delta^2} = \varepsilon\left(1+\underbrace{\frac{1}{2}\left(\frac{\Delta}{\epsilon}\right)^2-\frac{1}{8}\left(\frac{\Delta}{\epsilon}\right)^4}_{\delta E}\right) = \epsilon+\delta E$$

$$\ln 2ch\left(\frac{\beta E}{2}\right) = \ln 2ch\left(\frac{\beta \epsilon}{2}\right) + \frac{\beta}{2} tgh\left(\frac{\beta\epsilon}{2}\right)\delta E$$

$$+\left(\frac{\beta}{2}\right)^2 \frac{1}{2ch^2\left(\frac{\beta\epsilon}{2}\right)}\delta E$$

$\ln\left[2\,ch\left(\frac{x}{2}\right)\right]$

1) $\frac{1}{2ch\left(\frac{x}{2}\right)\cancel{\frac{1}{2}}}\, 2\,\frac{1}{2} sh\left(\frac{x}{2}\right) = \frac{1}{2} tgh\left(\frac{x}{2}\right)$ first derivative

2) $\frac{1}{2}\,\frac{1}{ch^2\left(\frac{x}{2}\right)}\,\frac{1}{2} = \frac{1}{4ch^2\left(\frac{x}{2}\right)}$ second derivative

$$\delta E = \frac{\Delta^2}{2\varepsilon} - \frac{\Delta^4}{8\epsilon^3}$$

$$\ln 2ch\,\frac{\beta E}{2} = \ln 2ch\left(\frac{\beta\epsilon}{2}\right) + \frac{\beta}{2} tgh\left(\frac{\beta\epsilon}{2}\right)\left[\frac{\Delta^2}{2\varepsilon} - \frac{\Delta^4}{8\epsilon^3}\right]$$

$$+\frac{1}{2}\left(\frac{\beta^2}{2^2}\right)\frac{1}{ch^2\left(\frac{\beta\epsilon}{2}\right)}\frac{\Delta^4}{4\epsilon^2}$$

$$\ln 2ch\,\frac{\beta E}{2} = \ln 2ch\left(\frac{\beta\epsilon}{2}\right) + \frac{\Delta^2}{4}\beta\,\frac{tgh\,\frac{\beta\epsilon}{2}}{\epsilon} +$$

$$+\frac{\beta}{8}\Delta^4\left[\frac{\beta^2}{4\,\epsilon^2 ch^2\left(\frac{\beta\epsilon}{2}\right)} - \frac{\beta}{2}\,\frac{tgh\left(\frac{\beta\epsilon}{2}\right)}{\epsilon^3}\right]$$

Contribution to Landau

$$-\frac{2}{\beta}\ln 2ch\,\frac{\beta\epsilon}{2} = -\frac{2}{\beta}\ln 2ch\,\frac{\beta\epsilon}{2}$$

$$-\frac{2}{\beta}\,\frac{\Delta^2}{4}\,\beta\,\frac{tgh\,\frac{\beta\epsilon}{2}}{\epsilon}$$

$$+\cancel{\frac{2}{\beta}}\,\frac{\beta}{8}\,\Delta^4\left[\cancel{\frac{\beta}{2}}\,\frac{tgh\,\frac{\beta\epsilon}{2}}{\epsilon^3}-\left(\frac{\beta}{2}\right)^{\cancel{2}}\frac{1}{\epsilon^2 ch^2(\frac{\beta\epsilon}{2})}\right]$$

$$\frac{\beta}{8}\,\Delta^4\underbrace{\left[\frac{tgh\,\frac{\beta\epsilon}{2}}{\epsilon^3}-\left(\frac{\beta}{2}\right)\frac{1}{\epsilon^2 ch^2(\frac{\beta\epsilon}{2})}\right]}_{(b)}$$

$$\underbrace{\frac{\Delta^2}{V_0}-\frac{\Delta^2}{2}\,\frac{th(\frac{\beta\epsilon}{2})}{\epsilon}}_{(a)}$$

$$\Psi(\Delta)=\Psi_0(\Delta)-\frac{2}{\beta}\left\langle \ln 2ch\,\frac{\beta\epsilon}{2}\right\rangle_F+a(T)|\Delta|^2+b(T)|\Delta|^4$$

$$a(T)=\frac{1}{V_0}-\frac{1}{2}\left\langle\frac{tgh(\beta\epsilon/2)}{\epsilon}\right\rangle_F$$

$$b(T)=\frac{\beta}{8}\left[\left\langle\frac{tgh(\frac{\beta\epsilon}{2})}{\epsilon^3}\right\rangle_F-\frac{\beta}{2}\left\langle\frac{1}{\epsilon^2 ch^2(\frac{\beta\epsilon}{2})}\right\rangle_F\right]>0$$

Gap Equation and T_c

$\frac{\partial \Psi}{\partial \Delta} = 0$ assume Δ real Ψ do not depend on phase

$$\frac{2\Delta}{V_0} - \frac{\cancel{2}}{\cancel{\beta}}\frac{\cancel{\beta}}{2}\left\langle \text{tgh}\frac{\beta E}{2}\frac{\partial E}{\partial \varepsilon}\right\rangle_F = 0$$

$$\frac{2\Delta}{V_0} - \left\langle \text{tgh}\frac{\beta E}{2}\frac{2\Delta}{2\sqrt{\epsilon^2+\Delta^2}}\right\rangle_F = 0$$

$$\boxed{\Delta = \frac{V_0}{2}\Delta\int_{-\epsilon_d}^{\epsilon_d} d\epsilon\, \mathcal{N}(\varepsilon)\,\frac{\text{tgh}\left[\frac{\beta E(\varepsilon)}{2}\right]}{E(\varepsilon)}}$$

↳ trivial $\Delta = 0$

↳ nontrivial

$$\boxed{1 = \frac{V_0}{2}\int_{-\epsilon_d}^{\epsilon_d} d\epsilon\, \mathcal{N}(\varepsilon)\,\frac{\text{tgh}\left(\frac{\beta E(\varepsilon)}{2}\right)}{E(\varepsilon)}}$$

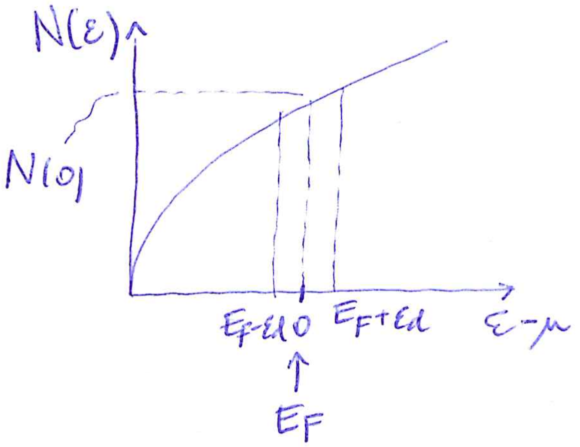
T_c is for $a(T_c) = 0$

$$1 = \frac{V_0}{2}\int_{-\epsilon_d}^{\epsilon_d} d\varepsilon\, \mathcal{N}(\varepsilon)\,\frac{\text{tgh}\,\frac{\beta|\epsilon|}{2}}{|\varepsilon|}$$

determines T_c

Tc

$$1 = \frac{V_0}{2}\int_{-\varepsilon_d}^{\varepsilon_d} d\varepsilon\, N(\varepsilon)\, \frac{tgh\left(\frac{\beta\varepsilon}{2}\right)}{\varepsilon}$$



around E_F if $\varepsilon_d \ll E_F$

$$N(\varepsilon) \simeq N(0)$$

$$1 = V_0 N(0)\int_0^{\varepsilon_d} d\varepsilon\, \frac{tgh\,\frac{\beta\varepsilon}{2}}{\varepsilon} = V_0 N(0)\int_0^{\frac{\beta\varepsilon_d}{2}} dx\, \frac{tgh(x)}{x}$$

as $\frac{\beta\varepsilon_d}{2} \gg 1$ integrand diverges logarithmically we can extract the explicit dependence on upper cutoff $\Lambda = \beta\frac{\varepsilon_d}{2}$ as

$$\int_0^{\Lambda} dx\, \frac{tgh(x)}{x} = \int_0^{\Lambda} d\ln x\; tgh(x) = \underbrace{\ln x\, tgh(x)\Big|_0^{\Lambda}}_{\text{diverges logarithmically}} - \underbrace{\int_0^{\Lambda} dx\, \frac{\ln x}{ch^2(x)}}_{\text{do not depend much on cutoff}}$$

$$\int_0^{\Lambda} dx\, \frac{tgh(x)}{x} = \ln\Lambda\, \underbrace{tgh(\Lambda)}_{\text{as } \Lambda \gg 1,\ \sim 1} - \underbrace{\int_0^{\infty} dx\, \frac{\ln x}{ch^2(x)}}_{\text{constant} = K}$$

$$1 \simeq V_0 N(0) \left[\ln \frac{\beta \epsilon_d}{2} - \mathbb{K} \right]$$

$$\frac{\epsilon_d}{k_B T_c} = 2 e^{\mathbb{K}} e^{\frac{1}{V_0 N(0)}}$$

$$k_B T_c = \frac{e^{-\mathbb{K}}}{2} \epsilon_d e^{-\frac{1}{V_0 N(0)}}$$

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$$\boxed{k_B T_c = 1{,}13\, \hbar \omega_d\, e^{-\frac{1}{V_0 N(0)}}}$$

a non perturbative (MFT) formula for T_c
(in V_0 !!)