$\boxed{\text{Bogolubov Diagonalization}}$

$$\varepsilon\left(c^\dagger_\uparrow c_\uparrow - c_\downarrow c^\dagger_\downarrow\right) - \left[\Delta^* c_\downarrow c_\uparrow + \Delta c^\dagger_\uparrow c^\dagger_\downarrow\right] = H$$

$$H^\dagger = \varepsilon\left(c^\dagger_\uparrow c_\uparrow - c_\downarrow c^\dagger_\downarrow\right) -$$

$$- \Delta c^\dagger_\uparrow c^\dagger_\downarrow - \Delta^* c^{\cancel{\dagger}}_\downarrow c^{\cancel{\dagger}}_\uparrow = H$$

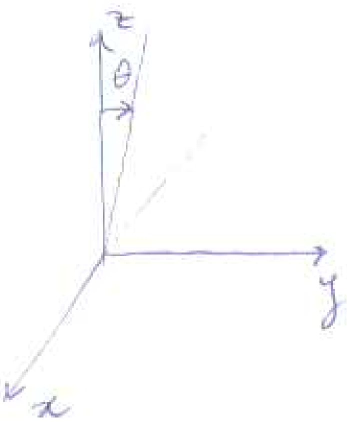
$$H = \left(c^\dagger_\uparrow \;\; c_\downarrow\right) \begin{bmatrix} \varepsilon & -\Delta \\ -\Delta^* & -\varepsilon \end{bmatrix} \begin{pmatrix} c_\uparrow \\ c^\dagger_\downarrow \end{pmatrix}$$

$$H_0 = \varepsilon \sigma_z - \mathrm{Re}\Delta\, \sigma_x + \mathrm{Im}\Delta\, \sigma_y$$

~~assume~~ eigenvalues

$$H_0 = \sqrt{\varepsilon^2 + |\Delta|^2}\, \vec{\sigma}\cdot\hat{n} \qquad E = \pm\sqrt{\epsilon^2 + |\Delta|^2} = E_\pm$$

eigenvectors assume Δ real



$$\theta = \mathrm{arctg}\,\frac{\Delta}{\epsilon}$$

in opposite direction wrt RHR

$$|\pm\rangle = \underbrace{e^{\mp\frac{i}{\hbar}\theta S_y}}\,|z\pm\rangle$$

$$e^{\mp\frac{i}{\hbar}\theta\frac{\hbar}{2}\sigma_y} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\sigma_y$$

$\langle \lambda' | H | \lambda \rangle$

$$\langle \alpha' | H | \alpha \rangle = \underbrace{\langle \alpha' | U^{\dagger}}_{\langle \lambda' |} \overbrace{U H U^{\dagger}}^{\text{diagonal}} \underbrace{U | \alpha \rangle}_{| \lambda \rangle}$$

$$(c^{\dagger}_{\uparrow} \; c_{\downarrow}) \, U^{\dagger} \; \underbrace{U \begin{bmatrix} \varepsilon & -\Delta \\ -\Delta & -\varepsilon \end{bmatrix} u^{\dagger}}_{\text{diagonal}} \, u \begin{pmatrix} c_{\uparrow} \\ c^{\dagger}_{\downarrow} \end{pmatrix}$$

$$u \begin{pmatrix} c_{\uparrow} \\ c^{\dagger}_{\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} c_{\uparrow} \\ c^{\dagger}_{\downarrow} \end{pmatrix} = \begin{pmatrix} \gamma_{+} \\ \gamma^{\dagger}_{-} \end{pmatrix}$$

$$\gamma_{+} = \cos\theta/2 \; c_{\uparrow} - \sin\frac{\theta}{2} \, c^{\dagger}_{\downarrow}$$

$$\gamma^{\dagger}_{-} = \sin\theta/2 \; c_{\uparrow} + \cos\frac{\theta}{2} \, c^{\dagger}_{\downarrow}$$

$$\gamma_{-} = \sin\theta/2 \; c^{\dagger}_{\uparrow} + \cos\frac{\theta}{2} \, c_{\downarrow}$$

they are fermions

$$\{\gamma_{+}, \gamma^{\dagger}_{+}\} = \cos^{2}\frac{\theta}{2} \{c_{\uparrow}, c^{\dagger}_{\uparrow}\} + \sin^{2}\frac{\theta}{2} \{c^{\dagger}_{\downarrow}, c_{\downarrow}\}$$
$$- \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cancel{\{c_{\uparrow}, c_{\downarrow}\}} - \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cancel{\{c^{\dagger}_{\downarrow} \, c^{\dagger}_{\uparrow}\}}$$
$$= \cos^{2}\frac{\theta}{2} + \sin^{2}\frac{\theta}{2} = 1$$

$$\{\gamma_{+}, \gamma^{\dagger}_{-}\} = \cos\frac{\theta}{2} \sin\frac{\theta}{2} \{c_{\uparrow} \, c_{\uparrow}\} - \sin\frac{\theta}{2} \cos\frac{\theta}{2} \{c^{\dagger}_{\downarrow} \, c^{\dagger}_{\downarrow}\} = 0$$

Hamiltonian in the new basis is

$$
\begin{pmatrix} \gamma_+^\dagger & \gamma_- \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \begin{pmatrix} \gamma_+ \\ \gamma_-^\dagger \end{pmatrix} =
$$

$$
= \begin{pmatrix} \gamma_+^\dagger & \gamma_- \end{pmatrix} \begin{pmatrix} E\gamma_+ \\ -E\gamma_-^\dagger \end{pmatrix} = E\,\gamma_+^\dagger \gamma_+ - E\,\gamma_- \gamma_-^\dagger
$$

It is possible to generalize to non real Δ as

$$
\begin{pmatrix} \gamma_+ \\ \gamma_-^\dagger \end{pmatrix} = \begin{bmatrix} u_k & -v_k \\ v_k^* & u_k^* \end{bmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix}
$$

where $|u_k|^2 + |v_k|^2 = 1$