Exact solution of the infinite range (or ∞ dimensional) Ising model

Consider ∞ range Ising model

$$H = -\frac{J}{2}\sum_{i\neq j}\sigma_i\sigma_j - h\sum_i \sigma_i$$

sum is extended to all i and all $j \neq i$
this term is no longer extensive
unless $J \to J/N$ indeed it consists
of $N(N-1) \sim N^2$ terms therefore
we define

$$H = -\frac{J}{2N}\sum_{i\neq j}\sigma_i\sigma_j - h\sum_i\sigma_i$$

as correct extensive Hamiltonian.

$$m = \frac{1}{N}\sum_{i=1}^{N}\sigma_i$$

$$m^2 = \frac{1}{N^2}\sum_{ij}\sigma_i\sigma_j = \frac{1}{N^2}\sum_i\sum_{j\neq i}\sigma_i\sigma_j + \frac{1}{N^2}\sum_i \sigma_i^2$$

$1/N$

$$H = -\frac{J}{2}(Nm^2+1) - hNm$$

linearize the m^2 term trough a gaussian transformation

$$e^{\frac{\beta J}{2}Nm^2} = \int \frac{dx}{\sqrt{2\pi}}\, e^{-x^2/2 + \sqrt{\beta J N}\, m x}$$

it's convenient to define a variable $\sqrt{\beta J N}\, x = \beta N y$
such as the exponent in the integral is extensive

$$e^{\frac{\beta J}{2}Nm^2} = \int \frac{dy}{\sqrt{2\pi J/\beta N}}\, e^{-\frac{\beta N}{2J}y^2 + \beta y N m}$$

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H} = \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi J/\beta N}}\, e^{-\frac{\beta N}{2J} y^2}\, e^{\frac{\beta J}{2}} \sum_{\{\sigma_i\}} e^{\beta N (y+h) \frac{\sum_i \sigma_i}{N}}$$

irrelevant

$$Z = e^{\beta J/2} \int \frac{dy}{\sqrt{2\pi J/\beta N}}\, e^{-\frac{\beta N}{2J} y^2} \left[2\,\mathrm{ch}\,(\beta(y+h))\right]^N$$

or in exponential form

$$Z = e^{\beta J/2} \int \frac{dy}{\sqrt{2\pi J/\beta N}}\, e^{-\beta \Psi(y)}\, \cancel{2^N}$$

$$\Psi(y) = N\left\{ \frac{y^2}{2J} - \frac{1}{\beta} \ln 2\,\mathrm{ch}\,[\beta(y+h)] \right\}$$

y has the meaning of local field due to interaction
it is therefore natural to change variable and
introduce a continuous magnetization
field m such as $y = Jm$

irrelevant

$$Z = e^{\beta J/2 \cancel{2N}} \int_{-\infty}^{+\infty} \sqrt{\frac{J\beta N}{2\pi}}\, dm\; e^{-\beta \Psi(m)}$$

$$\Psi(m) = N\left\{ \frac{J}{2} m^2 - \frac{1}{\beta} \ln 2\,\mathrm{ch}\,[\beta(Jm+h)] \right\}$$

the quantity

$$P(m) = \frac{e^{-\beta \Psi(m)}}{\int_{-\infty}^{+\infty} dm \, e^{-\beta \Psi(m)}}$$

can be interpreted as the distribution of the global magnetization

however a careful calculation shows that the exact distribution of m ~~is~~ approaches the one above when $m \simeq 0$

i.e. around the ~~the~~ critical point

$\hat{M} = \sum_i \sigma_i$

$\boxed{\text{Distribution of Magnetization in } \infty \text{ range Ising Model}}$

$$H = -\frac{J}{2N}\hat{M}^2 - h\hat{M} + \text{const}$$

$$P(M) = \frac{1}{Z}\,\mathrm{tr}\, e^{-\beta H(\hat{M})}\,\delta(M-\hat{M}) = \frac{e^{\frac{\beta J}{2N}M^2 + \beta h M}}{Z}\,\mathrm{tr}\,\delta(M-\hat{M})$$

$$\mathrm{tr}\,\delta(M-\hat{M}) = \frac{N!}{N_\uparrow!\,N_\downarrow!} \qquad \begin{array}{l} M = N_\uparrow - N_\downarrow \\ N = N_\uparrow + N_\downarrow \end{array}$$

$n^N e^{-N}$

$N\ln n - N$

$$\mathrm{tr}\,\delta(M-\hat{M}) = \frac{N!}{\left(\frac{N+M}{2}\right)!\left(\frac{N-M}{2}\right)!}$$

$$\ln \mathrm{tr}\,\delta(M-\hat{M}) = N[\ln(N) - 1] - \frac{M+N}{2}\left(\ln\left(\frac{M+N}{2}\right) - \cancel{\frac{M+N}{2}}^{1}\right)$$

$$-\left(\frac{N-M}{2}\right)\left(\ln\left(\frac{N-M}{2}\right) - \cancel{\frac{N-M}{2}}^{1}\right)$$

$$= N\ln N - \frac{M+N}{2}\ln\frac{M+N}{2} - \frac{N-M}{2}\ln\frac{N-M}{2} - \cancel{N} + \cancel{\frac{M+N}{2}} + \cancel{\frac{N-M}{2}}$$

$$\psi(M) = -\frac{J}{2N}M^2 - Mh - \frac{1}{\beta}\cancel{\ln}\left[N\ln N - \frac{M+N}{2}\ln\frac{M+N}{2} - \frac{N-M}{2}\ln\frac{N-M}{2}\right]$$

$$= N\left\{-\frac{J}{2}m^2 - mh - \frac{1}{\beta}\left[\ln N - \frac{1+m}{2}\ln N\left(\frac{1+m}{2}\right) - \frac{1-m}{2}\ln\frac{1-m}{2}N\right]\right.$$

$$\cancel{\ln N} - \frac{1+\cancel{m}}{2}\ln N - \cancel{\frac{1-m}{2}}\ln N$$

$$= N\left\{-\frac{J}{2}m^2 - mh + \frac{1}{\beta}\underbrace{\left[\frac{1+m}{2}\ln\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\ln\frac{1-m}{2}\right]}_{S(m)}\right\}$$

expand for small m

$$S(m) \simeq -\ln 2 + \frac{m^2}{2} + \frac{m^4}{12}$$

$$\Psi(m) = \underbrace{-N \ln 2}_{\Psi_0} + N\left\{ \left(\frac{1}{\beta} - J\right)\frac{m^2}{2} - mh + \frac{1}{\beta}\frac{m^4}{12} \right\}$$

$$\Psi(m) = \Psi_0 + \frac{NJ}{2}\left\{ \left(\frac{T}{T_c} - 1\right) m^2 + \frac{T}{T_c}\frac{m^4}{6} - m\frac{2h}{J} \right\}$$

compare with Landau $(h=0)$

$$\Psi(m) = N\frac{J}{2}\left\{ \underbrace{\left(1 - \frac{T_c}{T}\right)}_{\sim \frac{T-T_c}{T_c}} m^2 + \frac{1}{6}\underbrace{\left(\frac{T_c}{T}\right)^3}_{\sim 1} m^4 \right\}$$

and we see that actually the distribution of $\hat{M}/N$ approaches that obtained from Landau but it is not exactly the same