Ising Model (E Ising 1925)

a model for ferromagnetism that can be easily generalised to describe other kind of phase transitions as gas/liquid (The lattice Gas) antiferromagnetism etc

$$H = -\frac{J}{2}\sum_{\langle ij\rangle}\sigma_i\sigma_j - h\sum_i \sigma_i \qquad \sigma_i = \pm 1$$

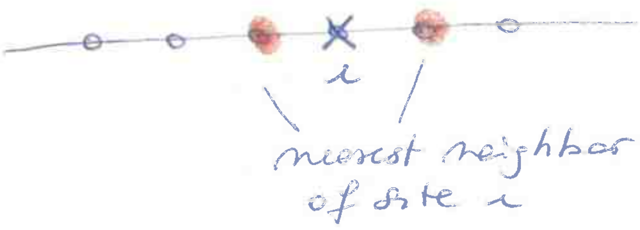
spin spin ferromagnetic ($J>0$) interaction

external field (constant)

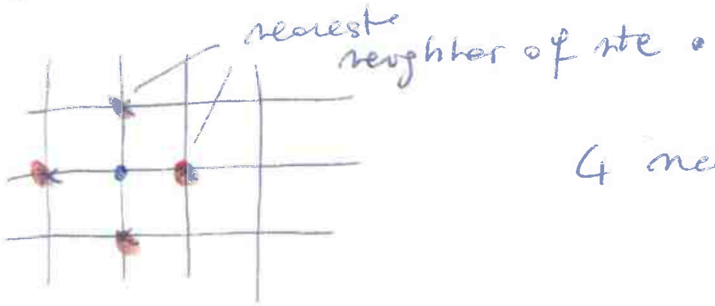
generalise to local ext fields $-\sum_i h_i\sigma_i$

Sum are on a lattice and $\sum_{\langle ij\rangle}$ means sum over nearest neighbor

$d=1$ linear

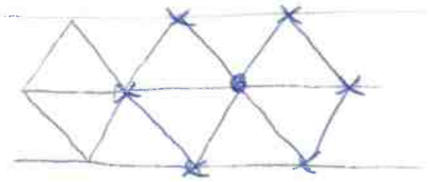


$d=2$ square

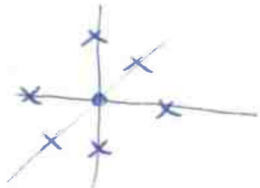


4 nearest neighbor

2d triangular



3d cubic



6 nearest neighbor

z is the # of nearest neighbor.

Ising model can be analytically solved in $d=1$
in $d=2$ square lattice (analytical for $h=0$)

or using approximate theory as the
mean field theory which amounts to
replace the interaction part by a single
spin hamiltonian

$$H \to \quad H_{eff} = -\cancel{J}\sum_i \sigma_i (h_{self} + h)$$

self-consistent
internal field

where

$$h_{self} = J \sum_j^{(n)} \langle \sigma_j \rangle = z J \langle \sigma \rangle$$

for
homogeneous ext field

Mean Field theory in Ising model by variational principle

Chose Heff as a single resoluble spin Hamiltonian

$$H_{eff} = -\sum_i h_i^{loc} \sigma_i$$

$$Z_{eff} = \prod_i 2\,ch(\beta h_i^{loc})$$

$$F_{eff} = -\frac{1}{\beta} \sum_i \ln 2\,ch(\beta h_i^{loc})$$

The function to be minimised is

$$\Phi(\{h_i^{loc}\}) = F_{eff} + \langle H - H_{eff} \rangle_{eff}$$

define

$\langle \sigma_i \rangle_{eff} = m_i$ and using Heff we get $m_i = tgh(\beta h_i^{loc})$

$$\langle H \rangle_{eff} = -\frac{J}{2} \sum_{\langle ij \rangle} m_i m_j - h \sum_i m_i$$

$$\langle H_{eff} \rangle_{eff} = -\sum_i h_i^{loc} m_i$$

because Heff is a single spin Ham. therefore

$$\langle \sigma_i \sigma_j \rangle_{eff} = \langle \sigma_i \rangle_{eff} \langle \sigma_j \rangle_{eff} \qquad i \neq j$$

$$\frac{\partial \varphi}{\partial h_i^{loc}} = 0 \Rightarrow \frac{\partial}{\partial h_i^{loc}}\left[\langle H\rangle_{eff} - \langle H_{eff}\rangle_{eff} + F_{eff}\right]$$

depends only on m — depends on m & h — depends only on h

$$\frac{\partial \langle H_{eff}\rangle_{eff}}{\partial h_i^{loc}} = \frac{\partial \langle H_{eff}\rangle_{eff}}{\partial m_i}\frac{\partial m_i}{\partial h_i^{loc}} + \left.\frac{\partial \langle H_{eff}\rangle}{\partial h_i^{loc}}\right|_{m=\text{cons}}$$

$-m_i$

$$\frac{\partial \langle H\rangle_{eff}}{\partial h_i^{loc}} = \frac{\partial \langle H\rangle_{eff}}{\partial m_i}\frac{\partial m_i}{\partial h_i^{loc}}$$

$$\frac{\partial F_{eff}}{\partial h_i^{loc}} = -m_i$$

$$\frac{\partial \varphi}{\partial h_i^{loc}} = \underbrace{\left(\frac{\partial \langle H\rangle_{eff}}{\partial m_i} - \frac{\partial \langle H_{eff}\rangle_{eff}}{\partial m_i}\right)}_{-J\Sigma_j^{(i)} m_j - h + h_i^{loc} = 0}\frac{\partial m_i}{\partial h_i^{loc}} + \cancel{m_i} - \cancel{m_i} = 0$$

$$\boxed{h_i^{loc} = J\Sigma_j^{(i)} m_j + h} \quad (1)$$

internal field (1)

interactions — ext field

which gives the self-consistency condition

$$\boxed{m_i = \tanh\left[\beta\left(J\Sigma_j^{(i)} m_j + h\right)\right]} \quad (2)$$

(Curie Weiss

And the free-energy in the mean-field approximation

$$\Phi = \underbrace{-\frac{J}{2}\sum_{\langle ij\rangle} m_i m_j - h \cancel{\sum_i} m_i}_{\langle H\rangle_{eff}} + \underbrace{\sum_i h_i^{loc} m_i}_{-\langle H_{eff}\rangle_{eff}} \underbrace{- \frac{1}{\beta}\sum_i \ln 2ch\, \beta h_i^{loc}}_{F_{eff}}$$

we can express h_i^{loc} trough m_i using (1)

$$\sum_i h_i^{loc} m_i = J \sum_{\langle ij\rangle} m_i m_j + h \cancel{\sum_i} m_i$$

$$\boxed{\Phi(\{m_i\}) = \frac{J}{2}\sum_{\langle ij\rangle} m_i m_j - \frac{1}{\beta}\sum_i \ln 2ch\, \beta\left[J \sum_j^{(i)} m_j + h\right]}$$

for homogeneous h the Ising Hamiltonian is lattice-translation invariant

if a lattice translation by lattice vector R

T_R such that $\sigma_{i+R} = T_R \sigma_i T_R^+$

then

$$T_R H T_R^+ = -\frac{J}{2}\sum_{\langle ij\rangle} T_R \sigma_i \sigma_j T_R^+ - h \sum_i T_R \sigma_i T_R^+$$

$$= -\frac{J}{2}\sum_{\langle ij\rangle} \sigma_{i+R}\sigma_{j+R} - h\sum_i \sigma_{i+j} =$$

$$= -\frac{J}{2}\sum_{\langle i'j'\rangle} \sigma_{i'}\sigma_{j'} - h\sum_{i'} \sigma_{i'} = H$$

Notice that a local ext field break this symmetry

$T_R \sum_i h_i \sigma_i T_R^+ = \sum_i h_i \sigma_{i+R}$ is not invariant

operates only on σ_i

Then a generic average of local operator

$$\langle A_i \rangle = \frac{\operatorname{tr} e^{-\beta H} A_i}{\operatorname{tr} e^{-\beta H}} = \frac{1}{Z} \operatorname{tr} e^{-\beta H} T_R^+ T_R A_i$$

but since H is translation invariant

$$[H, T_R] = 0$$

and

$$[e^{-\beta H}, T_R^+] = 0$$

$$\langle A_i \rangle = \frac{1}{Z} \operatorname{tr} T_R^+ e^{-\beta H} T_R A_i = \frac{1}{Z} \operatorname{tr} e^{-\beta H} T_R A_i T_R^+$$

cyclic property of trace

$\langle A_i \rangle = \langle A_{i+R} \rangle \quad \forall R$ then $\langle A_i \rangle$ independ for a generic correlation function

$$C_{ij} = \langle A_i B_j \rangle = \frac{1}{Z} \operatorname{tr} e^{-\beta H} \underset{T_R^\dagger T_R}{\uparrow} A_i \underset{T_R^\dagger T_R}{\uparrow} B_j = \langle A_{i+R} B_{j+R} \rangle$$

therefore

$$C_{ij} = C_{i+R,\, j+R} \quad \forall R \Rightarrow C_{ij} = C(i-j)$$

for homogeneous $h_i = h$ the Curie-Weiss equation gives

$$\sum_j^{(i)} \langle \sigma_j \rangle_{eff} = \underset{\text{\# of nearest neighbors}}{z}\, m \qquad \text{since } m_i = \langle \sigma_i \rangle_{eff} = m$$

$$\boxed{m = \operatorname{tgh}[\beta(Jzm + h)]}$$

and

$$\boxed{G = N\left\{ \frac{J}{2} z m^2 - \frac{1}{\beta} \ln 2 \operatorname{ch}[\beta(Jzm + h)] \right\}}$$

It is worth to note that the same result can be obtained by linearizing the quadratic interaction

$$-\frac{J}{2}\sum_{ij}\sigma_i\sigma_j$$

formally $\sigma_i = m_i + \underbrace{(\sigma_i - m_i)}_{\delta\sigma_i}$

expanding the Hamiltonian to first order in $\delta\sigma$

$$H^{lin}(\sigma) = H(m) + \frac{\partial H}{\partial \underline{m}}\cdot\delta\underline{\sigma} + \cancel{O(\delta\sigma^2)}$$

we get

$$H^{lin}_{\cancel{eff}} = -\frac{J}{2}\sum_{\langle ij\rangle} m_i m_j - h\sum_i m_i +$$

$$+ J\sum_{\langle ij\rangle} m_j\,\delta\sigma_i \quad - h\sum_i\delta\sigma_i$$

now replace $\delta\sigma_i = \sigma_i - m_i$

$$H^{lin}_{\cancel{eff}} = -\frac{J}{2}\sum_{\langle ij\rangle} m_i m_j - \cancel{h\sum m_i} - J\sum_{\langle ij\rangle} m_j\sigma_i + J\sum_{\langle ij\rangle} m_i m_j$$

$$-h\sum_i\sigma_i + \cancel{h\sum m_i} = \frac{J}{2}\sum_{\langle ij\rangle} m_i m_j - \sum_i\Big(J\sum_j^{(i)} m_j + h\Big)\sigma_i$$

$$\varphi^{\text{eon}} = -\frac{1}{\beta} \ln Z^{\text{eon}}$$

$$Z^{\text{eon}} = \operatorname{tr} e^{-\beta H^{\text{eon}}}$$

$$\varphi^{\text{eon}} = \frac{J}{2} \sum_{\langle ij \rangle} m_i m_j - \frac{1}{\beta} \sum_i \ln 2\operatorname{ch}[\beta(J\Sigma_j^{(i)} m_j + h)]$$

therefore the MFT can be seen as neglecting deviations from mean value in the internal field

replacing

$$h_i^{int} = + J \Sigma_j^{(i)} \sigma_j \rightarrow J \Sigma_j^{(i)} m_i$$