Fourier trasform on a lattice

$d=1$

let f defined on a lattice

with periodic boundary condition

$$f(N)=f(0) \quad \text{i.e}$$

$$f(n+N) = f(n)$$

we can define

$$|R_n\rangle = \sum_k \langle k|R_n\rangle |k\rangle$$

$$|k\rangle = \sum_{R_n} \langle R_n|k\rangle |R_n\rangle$$

where $f(n) = \langle R_n| f\rangle$

$\qquad f(k) = \langle k| f\rangle$

and both R_n an k are discrete

$$\langle R_n|k\rangle = \frac{1}{\sqrt{N}} e^{ikR_n} \qquad \langle k|R_n\rangle = \frac{1}{\sqrt{N}} e^{-ikR_n}$$

for that the completness relation must hold

$$\sum_{R_n} \langle k|R_n\rangle\langle R_n|k'\rangle = \delta_{k,k'} \quad \text{Kronecker}$$

that is

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{-ikR_n} e^{ik'R_n} =$$

$$\frac{1}{N}\sum_{m=0}^{N-1} e^{i(k'-k)ma} = \delta_{k,k'}$$

by implementing the periodic boundary condition

$$f(m+N) = f(m)$$

$$|f\rangle = \sum_k \langle k|f\rangle |k\rangle$$

$$|f\rangle = \sum_R \langle R|f\rangle |R\rangle$$

$$\langle m|f\rangle = \sum_k \langle k|f\rangle \langle m|k\rangle =$$

$$= \frac{1}{\sqrt{N}}\sum_k e^{ikRm} f(k)$$

$$f(m+N) = \frac{1}{\sqrt{N}}\sum_k e^{ik(n+N)a} f(k)$$

$$e^{ikNa} = e^{im2\pi}$$

$$\boxed{k_m = m\frac{2\pi}{Na}}$$ periodic boundary condition

ma is discrete and is also distributed on a finite set of N states.

To fulfill the completeness

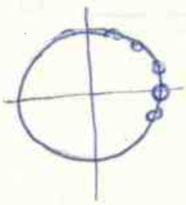
$$\frac{1}{N}\sum_{m=0}^{N-1} e^{ina(m'-m)\frac{2\pi}{Na}} \qquad |m'-m| \leq N$$

finite spacings lattice discretization

$$\boxed{m = -\frac{N}{2}+1, \frac{N}{2}}$$ such that $(m'-m)_{max} = \frac{N}{2} - \left(-\frac{N}{2}+1\right) = N-1$

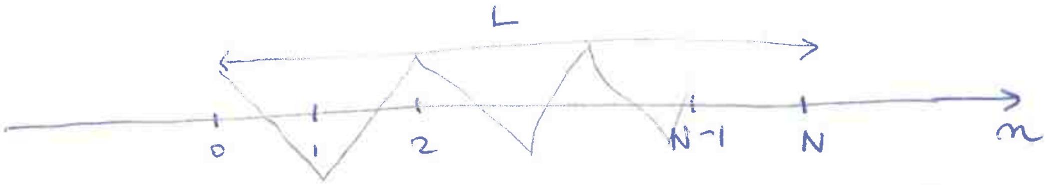
for any fixed $m'-m$ integer $(m'-m)_{min} = -\frac{N}{2}+1-\frac{N}{2} = -N+1$

$\neq 0$



$$f(k_{max}) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-i \frac{N}{\cancel{2}} \frac{2\pi}{N\cancel{a}} n\cancel{a}} f(na)$$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{i\pi n a} f(na) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (-1)^n f(na)$$



this represents the ~~maximum~~ minimum wave lenght on finite spacings lattice

$$\Delta k = \frac{2\pi}{Na} \qquad K_{max} = \frac{\cancel{2}\pi}{\cancel{N}a} \frac{\cancel{N}}{\cancel{2}} = \frac{\pi}{a} \qquad \lambda = 2a$$

to resume

$$\boxed{\begin{aligned} f(n) &= \sum_{k_m=-\frac{N}{2}+1}^{N/2} \frac{1}{\sqrt{N}} e^{ik_m \cdot na} f(k_m) \\ f(k_m) &= \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{-ik_m na} f(na) \end{aligned}}$$

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{i(k'-k)na} = \delta_{k,k'} = \delta_{m,m'}$$

$$\frac{1}{N} \sum_{k_m=-N/2+1}^{N/2} e^{ik_m(n-n')a} = \delta_{n,n'}$$