$$\Psi(m) = N\left\{\frac{1}{2} J z m^2 - \frac{1}{\beta} \ln 2\mathrm{ch}[\beta(Jzm+h)]\right\}$$

(a)

Mean Field & Landau Theory

$\downarrow$ minimum $\qquad m = \langle \sigma_i \rangle_{eff}$

$$m = \mathrm{tgh}\,[\beta(Jzm+h)]$$

$\downarrow$ Solution

$$\mathrm{tgh}^{-1}(m) = \beta(Jzm+h)$$

$$\frac{1}{\beta}\mathrm{tgh}^{-1}(m) = Jzm+h \qquad \cancel{m=\frac{1}{Jz}}$$

$$h = \frac{1}{\beta}\mathrm{tgh}^{-1}(m) - Jzm = \frac{1}{\beta}\frac{1}{2}\ln\left(\frac{1+m}{1-m}\right) - Jzm$$

$$\frac{\partial h}{\partial m} = \frac{1}{2\beta}\left[\frac{1}{1+m} + \frac{1}{1-m}\right] - Jz$$

$$\frac{\partial h}{\partial m} = \frac{1}{\beta}\,\frac{1}{1-m^2} - Jz \qquad \left(\frac{\partial h}{\partial m}\right)_{m=0} = \frac{1}{\beta} - Jz \begin{cases} >0 & \frac{1}{\beta} > Jz \\ <0 & \frac{1}{\beta} < Jz \end{cases}$$

$$\left(\frac{\partial m}{\partial h}\right) = \chi = \frac{1}{\frac{1}{\beta}\frac{1}{1-m^2} - Jz} = \frac{\beta}{\frac{1}{1-m^2} - \beta Jz} \qquad k_B T_c = Jz$$

$$\beta Jz = \frac{T_c}{T}$$

$$\boxed{\chi = \frac{\beta(1-m^2)}{1-\beta Jz(1-m^2)}}$$

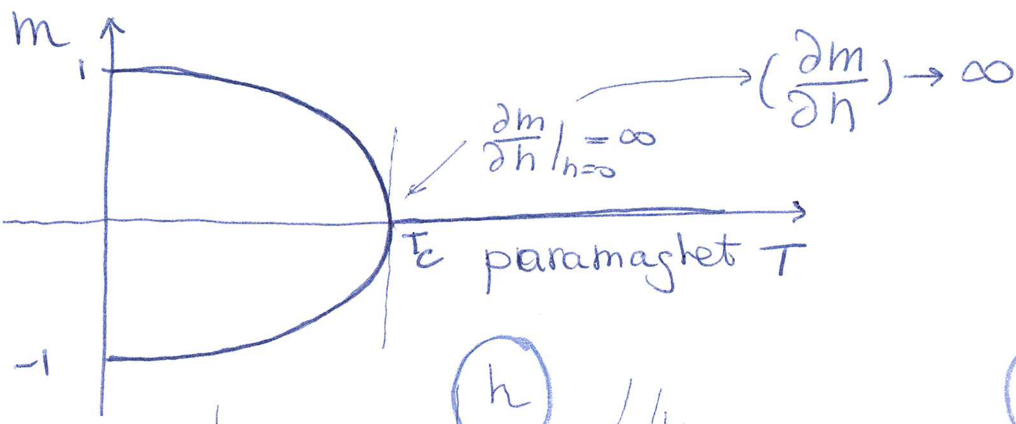
$\longrightarrow$ ~~Critical~~ 2nd order phase transition at T_c

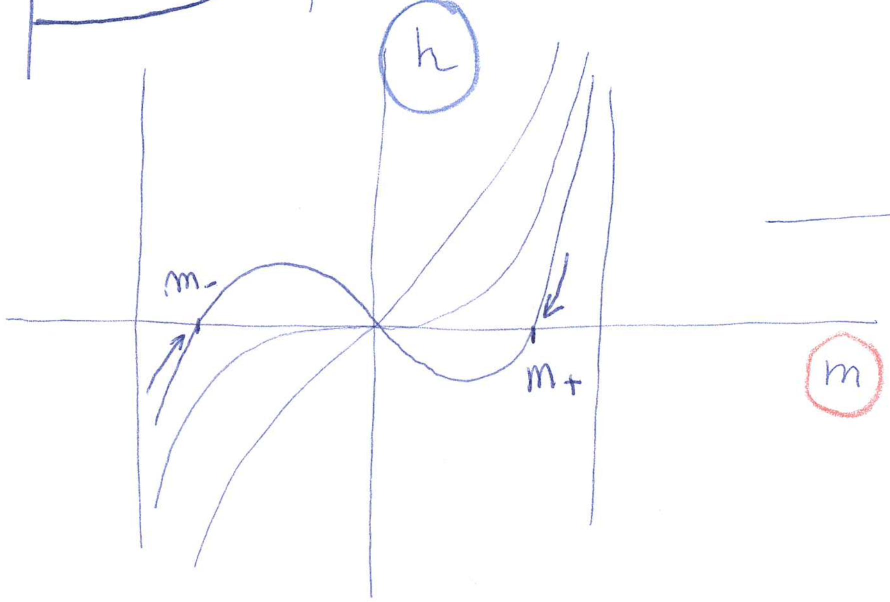
$\longrightarrow$ 1st order phase transition below T_c

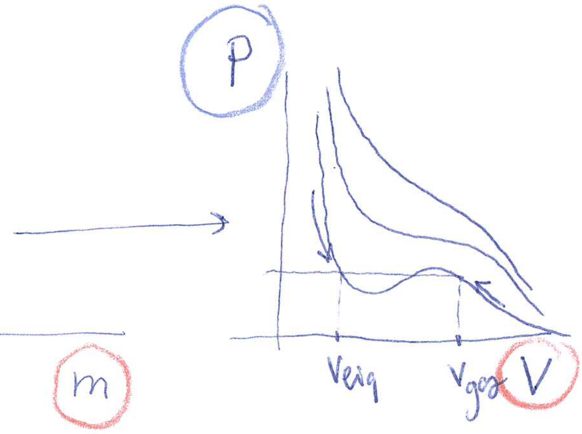
b

2nd order phase transition

$h=0 \quad \chi(T, h=0) = \infty$ at $T=T_c$







$$m = \frac{tr\, e^{-\beta H} \sigma}{tr\, e^{-\beta H}} = \frac{tr\, e^{+\beta J \sum_{ij} \sigma_i \sigma_j + \beta h \sum_i \sigma_i} \sigma_i}{tr\, e^{-\beta H}}$$

$$m = -\frac{1}{N} \frac{\partial F}{\partial h} \qquad \left(F = -\frac{1}{\beta} \ln tr\, e^{+\beta J \sum_{ij} \sigma_i \sigma_j + \beta h \sum_i \sigma_i} \right)$$

$$\frac{\partial m}{\partial h} = -\frac{1}{N} \frac{\partial^2 F}{\partial h^2} \leftarrow \text{2nd derivative diverges!}$$

1st order phase transition

(c)

$T < T_c$ domains of $m > 0$

coexisting with domains of $m < 0$

if $h = 0$

on the other hand if $T < T_c$

$$\lim_{h \to 0^+} m = m_+$$

$$\lim_{h \to 0^-} m = m_-$$

Spontaneous Symmetry breaking

the Thermodynamic state has less symmetries than the original H

Breaking of a "theorem"

when $h = 0$

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

discrete symmetry $\sigma_i \to -\sigma_i$ $\quad H \to \bar{H} = H$

Consider the operation $P \mid P^{-1} f(\sigma) P = f(-\sigma)$

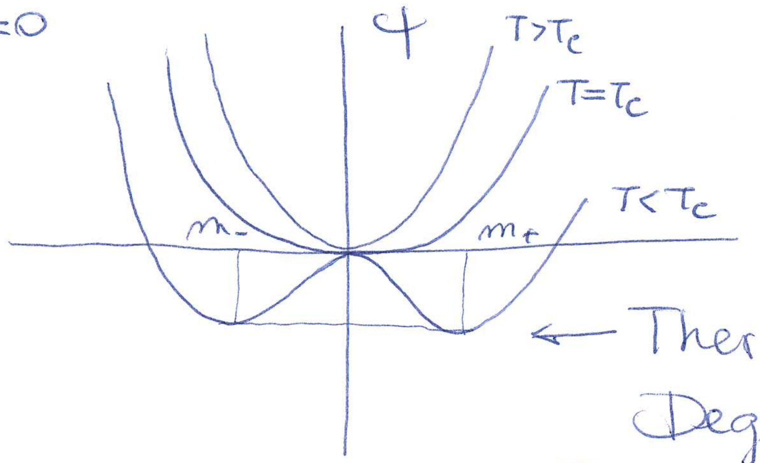
Consider

$$\langle P^{-1} \sigma_i P \rangle = \frac{\mathrm{tr}\, e^{-\beta H} P^{-1} \sigma_i P}{\mathrm{tr}\, e^{-\beta H}} \overset{\text{cyclic}}{=} \frac{\mathrm{tr}\, P e^{-\beta H} P^{-1} \sigma_i}{\mathrm{tr}\, e^{-\beta H}}$$

$$= \frac{\mathrm{tr}\, e^{-\beta \bar{H}} \sigma_i}{\mathrm{tr}\, e^{-\beta H}} = \frac{\mathrm{tr}\, e^{-\beta H} \sigma_i}{\mathrm{tr}\, e^{-\beta H}} = \langle \sigma_i \rangle$$

but $P^{-1} \sigma P = -\sigma \Rightarrow -\langle \sigma_i \rangle = \langle \sigma_i \rangle \Rightarrow \langle \sigma_i \rangle = 0$

$h=0$ (d)

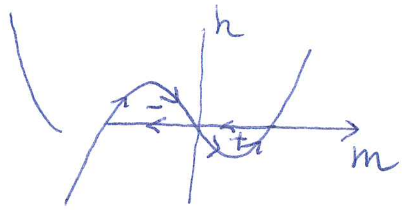


← Thermodynamically Degenerate States

Same Free Energy !

Maxwell construction

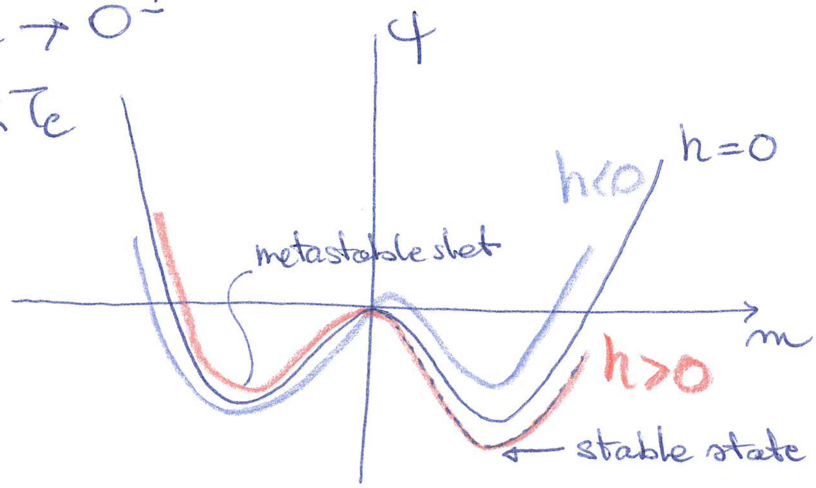
↳ coexistence



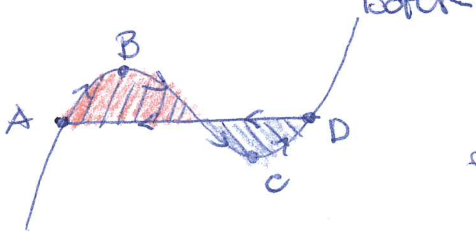
$\int dm\, h$

$h \to 0^{\pm}$

$T<T_c$



Maxwell's construction



work on the cycle

$$\underbrace{W}_{\text{extensive}} = -\underbrace{\oint \vec{h}\, d\vec{M}}_{} = -S_+ + S_-$$

total magnetiz.

$$dU = TdS + hdM$$

$$\oint dU = \underbrace{T\oint dS}_{\text{isothermal}} + \oint h\, dM$$

$\oint dU = 0$ state function

$\Downarrow$

$\oint h\, dM = 0$

$\Downarrow$

$S_+ = S_-$

Lattice gas and Universality

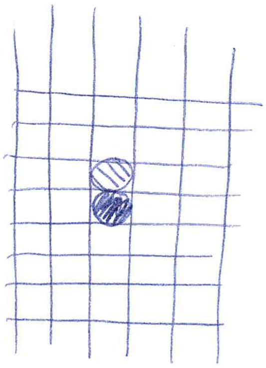
The same scenario arise in a model for gas/liquid transition in MEAN FIELD APPROX

The Lattice Gas

$$H-\mu N = -\frac{\phi}{2}\sum_{\langle\alpha,\beta\rangle} n_\alpha n_\beta - \mu \sum_\alpha n_\alpha \qquad \alpha = 1, M$$

M # of cells of lattice

If v_0 is the volume of one cell then $V = M v_0$



$n_\alpha = 0, 1$

Hard sphers on a lattice (Repulsive pot)

$\phi > 0 \Rightarrow$ Attractive potential

One can map the Lattice Gas onto the Ising model by taking $N = M$ spins and introducing the spin variable

$$\sigma_\alpha = 2 n_\alpha - 1 \quad \begin{cases} n_\alpha = 0 & \sigma_\alpha = -1 \\ n_\alpha = 1 & \sigma_\alpha = 1 \end{cases}$$

cell full $\Rightarrow$ up spin

all empty $\Rightarrow$ down spin

liq high density $\longrightarrow$ magnetization $\uparrow$

gas low density $\longrightarrow$ magnetization $\downarrow$

Exercise

f

a) proof that

H_{LG} maps onto

$$H_{Ising} - \frac{\phi}{8}\mathcal{M}z - \mu\frac{\mathcal{M}}{2}$$

when I consider $J = \phi/4$ Feromagn Interaction ⟷ Attracta

$h = \phi\frac{z}{4} + \frac{\mu}{2}$ Ext magn field ⟷ Chemical poten

b) recover the equation of state of lattice gas from that of the ising model ($h(m)$)

let $N = \Sigma_\alpha \langle n_\alpha \rangle$ total # of particle

then $N = \mathcal{M}\langle n \rangle$ for homogeneous system

If v_0 is the cell volume

$$\rho = \frac{N}{V} = \frac{\cancel{\mathcal{M}}\langle n \rangle}{\cancel{\mathcal{M}} v_0} \qquad \langle n \rangle = \rho v_0$$

density

$$P v_0 = -\frac{\phi z}{2}\langle n \rangle^2 + \frac{1}{\beta}\ln\frac{1}{1-\langle n \rangle}$$

attraction

repulsion

- Plot the pressure as a function of $\frac{1}{\langle n \rangle} = \frac{V}{N v_0} = \frac{V}{V_0}$

excluded volume

- determine T_c
- What is the divergent susceptibility?

2nd order phase transition
emergence of symmetry breaking
emergence of non zero order parameter
ORDER PARAMETER (m)

$m=0$ disordered (symmetric) phase

$m\neq 0$ ordered (Symmetry broken) phase

m is <u>Continuous</u> across the 2nd order phase transition

→ Developing ψ for small m
→ Universal aplies for all cases of <u>Scalar</u> order parameter

$$\boxed{\psi(m) = N[am^2 + bm^4 - hm]}$$

$$\boxed{\begin{array}{l} a \propto T - T_c \\ b > 0 \end{array}}$$

← Universal properties

Calculation in the Ising model

4

$$\Psi = N\left\{ \frac{1}{2} Jzm^2 - \frac{1}{\beta} \ln 2\mathrm{ch}[\beta(Jzm+h)] \right\}$$

$$\ln 2\mathrm{ch}(x) = \ln 2 + \ln \mathrm{ch}(x)$$

$$\ln \mathrm{ch}(x) \simeq \ln\Big(1 + \underbrace{\frac{x^2}{2} + \frac{x^4}{4!}}_{\varepsilon} + \dots\Big) =$$

$$= \cancel{\frac{x^2}{2}} \; \varepsilon - \frac{\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3) = \frac{x^2}{2} + \frac{x^4}{4!} - \frac{1}{2}\left(\frac{x^2}{2} + \frac{x^4}{4!}\right)^2 \simeq$$

$$\simeq \frac{x^2}{2} + \frac{x^4}{4!} - \frac{1}{2}\left(\frac{x^4}{4}\right) = \frac{x^2}{2} - \frac{x^4}{12}$$

$$\Psi(m) \simeq N\left\{ \frac{1}{2} Jzm^2 - \frac{1}{\beta}\left[\frac{[\beta(Jzm+h)]^2}{2} - \frac{[\beta(Jzm+h)]^4}{12} \right] \right\}$$

Consider the $h=0$ case

$$\Psi(m) \cong N\left\{ \frac{Jzm^2}{2}(1 - \beta Jz) + \frac{\beta^3}{12}(Jzm)^4 \right\}$$

remember

$$\beta Jz = \frac{T_c}{T}$$

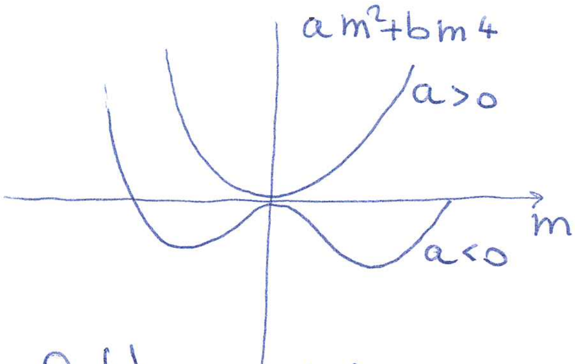
$$\boxed{\Psi(m, h=0) \cong N \frac{Jz}{2}\left\{ \left(1 - \frac{T_c}{T}\right) m^2 + \frac{1}{6}\left(\frac{T_c}{T}\right)^3 m^4 + \mathcal{O}(m^6) \right\}}$$

for $\dfrac{|T - T_c|}{T_c} \ll 1$

$$a = \frac{Jz}{2T_c}(T - T_c) \simeq \frac{k_B}{2}(T - T_c) \begin{cases} T > T_c & > 0 \\ T < T_c & < 0 \end{cases}$$

$$b \simeq \frac{Jz}{12} > 0$$

$\boxed{h=0}$



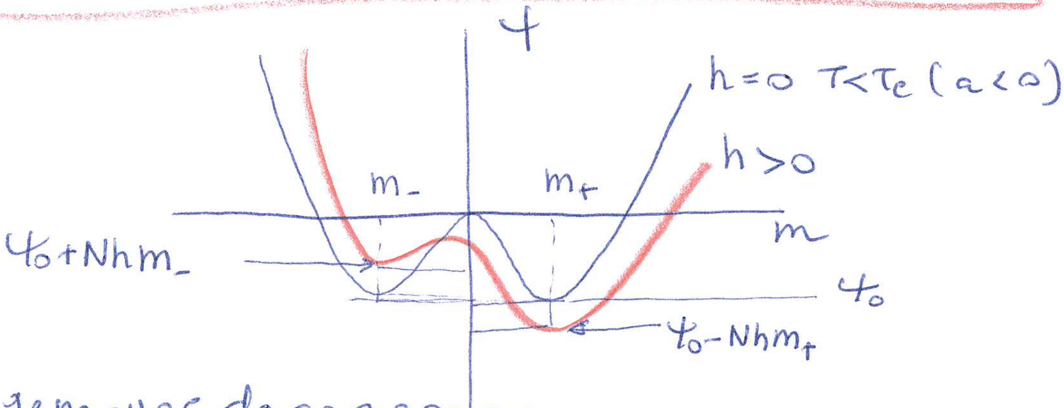
When $h \neq 0$ let's consider small h then

$$\Psi(m,h) = \Psi(m,h=0) + \left(\frac{\partial \Psi}{\partial h}\right)_{h=0} h$$

but $\frac{\partial \Psi}{\partial h} = \frac{\partial F_{MeanField}}{\partial h} = -Nm$

at self consistency

$$\boxed{\Psi(m,h) = N\{am^2 + bm^4 - hm\}}$$



h removes degeneracy therefore

$$\lim_{h\to 0^+} m(h) = m_+$$

$$\lim_{h\to 0^-} m(h) = m_-$$

(i)

Critical indexes in Landau theory

Universality of 2nd order p.t
→ measured by the same critical behaviour
i.e. the same beaviour of ~~crit~~ thermodyn
function around T_c

$$A = k(T-T_c)^{x_a}$$ critical index

$\boxed{\text{Classical (Landau) critical indexes}}$

$h=0$ ~~m~~ order parameter

Landau Curie-Weiss

$$\Phi = N\{am^2 + bm^4 - hm\}$$

$$\frac{\partial \Phi}{\partial m} = N\{2am + 4bm^3 - h\} = 0 \Rightarrow \bar{m}$$ variational

$h=0$

$(2a-4bm^2)m = 0$ Landau Curie Weiss

2 sols

$\to m = 0 \quad (a>0)$ ~~PARA~~ DISORDERED

$\to m^2 = -\frac{a}{2b} \quad (a<0)$ ORDERED

Stability

$\Phi_{DIS} = 0 \qquad \Phi_{ORD} = N\{-\cancel{\frac{a^2}{2b}}\cancel{+\frac{1}{4b}}\,a + b(-\frac{a}{2b})\}(-\frac{a}{2b}) = -\frac{a^2}{4b}$

$T<T_c \qquad m = \sqrt{-\frac{a}{2b}} \propto (T_c - T)^{1/2}$ $\quad$ m

$\beta = 1/2$ order param index β

$h=0$ susceptibility

Landau Curie-Weiss

$$2am + 4bm^3 - h = 0$$

$\chi = \frac{\partial m}{\partial h}$ susceptibility for spon

$$2a\chi + 12bm^2\chi - 1 = 0$$

$$\chi = \frac{1}{2a + 12bm^2}$$

<u>2 solutions</u>

$T>T_c$

$$\chi = \frac{1}{2a}$$

$T<T_c$

$$\chi = \frac{1}{2a - 12b\frac{a}{2b}} = -\frac{1}{4a}$$

$$\left.\right\} \frac{1}{|T-T_c|^{\gamma}}$$

$\gamma = 1$ susceptibility exponent γ

Critical isothermal ext field

$T = T_c \to a = 0$

Landau CW

$4bm^3 = h$

$h_{cr} \propto m^\delta$

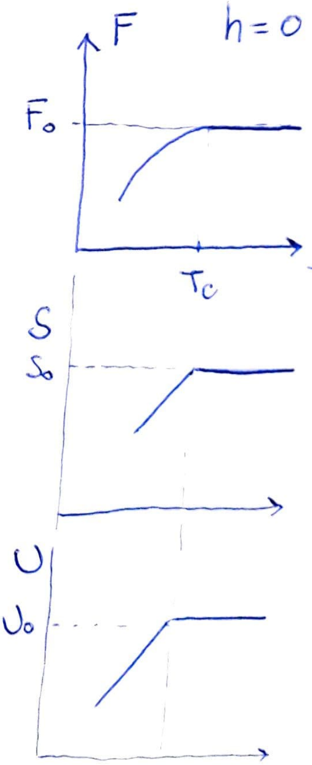
Critical field exponent $\delta = 1/3$

Specific heat

$F = \Psi(\bar{m}(h,T))$ $\bar{m}$ solution of Curie-Weiss

some constant that can be T dependent but not on $T_c - T$

$F = N\left\{a\left(-\frac{a}{2b}\right) + b\left(-\frac{a}{2b}\right)^2\right\} + F_0$ as $h = 0$



$T < T_c$ $m^2 = -a/2b$ $\quad$ $T > T_c$ $m^2 = 0$

$F = F_0 - \underbrace{\frac{a^2}{4b}}_{\Delta F}$ $\quad$ $F = F_0$

$S = -\frac{\partial F}{\partial T} = S_0 + \frac{\partial \Delta F}{\partial T}$ linear in $T_c - T$

$U = F + TS = F_0 - \underbrace{\cancel{\Delta F}}_{O((T-T_c)^2)} + T_c\left(S_0 + \frac{\partial \Delta F}{\partial T}\right)$ linear

$U = U_0 + T_c \frac{\partial \Delta F}{\partial T}$

$C_h = \left(\frac{\partial U}{\partial T}\right)_{h=0} \sim \frac{1}{(T-T_c)^\alpha}$

$\alpha = 0$