Mean-Field Susceptibility and critical exponent ν

let's consider non homogeneous field h_i then

$$m_i = tgh\left[\beta\left(\Sigma_K^{(i)} J_{iK} m_K + h_i\right)\right]$$

recall linear response $\chi_{ij} = \beta C_{ij}$

$$\chi_{ij} = \frac{\partial m_i}{\partial h_j}$$ non local susceptibility but evaluated in the homogeneous limit

$$\chi_{ij} = \frac{1}{ch^2(\dots)} \beta\left[\Sigma_K^{(i)} J_{iK} \chi_{Kj} + \delta_{ij}\right]$$

now consider $h_i = h \ \forall i \Rightarrow$ homogeneous limit

then $ch^2(\dots) = ch^2[\beta(Jzm+h)]$

and from

$ch^2 - sh^2 = 1 \qquad 1 - tgh^2 = \frac{1}{ch^2}$

then

$$\chi_{ij} = \beta(1-m^2)\left[\Sigma_K^{(i)} J_{iK} \chi_{Kj} + \delta_{ij}\right]$$

homogeneous limit

as homogeneous limit is concerned

$$\chi_{ij} = \chi(R_i - R_j)$$

lattice site

lattice Fourier transform

$$f(\vec{R}_l) = \frac{1}{N} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{R}_l} f(\vec{k})$$

$$f(\vec{k}) = \sum_R e^{-i\vec{k}\cdot\vec{R}} f(\vec{R})$$

R is defined on a lattice

both J_{ik} and X_{kj} depends only on distance

$$\sum_k^{(1)} J_{ik} X_{kj} = \overbrace{\sum_{R'} J(R-R') X(R'-R'')}^{\text{convolution}} =$$

$$= \frac{1}{N^2} \sum_{\substack{kk' \\ R'}} e^{ik(R-R')} e^{ik'(R'-R'')} J(k) X(k') =$$

$$= \frac{1}{N^2} \sum_{kk'} e^{ikR} e^{-ik'R''} J(k) X(k') \underbrace{\sum_{R'} e^{i(k'-k)R'}}_{N\delta_{k,k'}}$$

$$= \frac{1}{N} \sum_k e^{ik(R-R'')} J(k) X(k)$$

fourier transform of a convolution

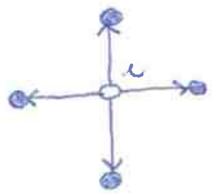
then by fourier transform of a lattice

$$X(\vec{k}) = \beta(1-m^2)[J(k)X(k) + 1]$$

Fourier transform of a Kronecker delta

$\sum_R e^{-ikR} \delta_R = 1$

$$J(\vec{k}) = J \sum_{\vec{\delta}} e^{-i\vec{k}\cdot\vec{\delta}}$$



$$\mathcal{J}(\vec{k}) = 2J \sum_{1}^{d}{}_{\alpha} \cos(k_\alpha a) \qquad \text{(hipercubic lattice)}$$

for small k expand

$$\mathcal{J}(\vec{k}) = 2J \sum_{1}^{d}{}_{\alpha}\, 1 - \frac{k_\alpha^2 a^2}{2} = zJ - Ja^2|k|^2$$

$$\chi(k)\left(1+\beta(1-m^2)\mathcal{J}(k)\right) = \beta(1-m^2)$$

$$\chi(k) = \frac{\beta(1-m^2)}{1-\beta(1-m^2)\mathcal{J}(k)} = \frac{\beta(1-m^2)}{1-\beta(1-m^2)zJ + \beta(1-m^2)Ja^2|k|^2}$$

$$\chi(k) = \frac{1}{\frac{1-\beta(1-m^2)zJ}{\beta(1-m^2)} + Ja^2|k|^2}$$

$\chi_0^{-1} \leftarrow$ thermodynamic susceptibility at $k=0$

$$\chi(\vec{k}) = \frac{\chi_0}{1+\chi_0 Ja^2|k|^2} = \frac{\chi_0}{1+\xi_c^2|k|^2}$$

Ornstein-Zernike type Correlation Function

ξ_c is the correlation length

$$\xi_c^2 = \chi_0 Ja^2 \qquad \xi_c \propto |T-T_c|^{-\nu}$$

$\nu = 1/2$ diverges at critical point! long-range correlations but performing the calculations within single site effective Hamiltonian (4)

$$H_{eff} = -\sum_i h_i^{loc} \sigma_i$$

$\Rightarrow$ all sites have independent distribution

$$P(\{\sigma_i\}) = \prod_i^N P(\sigma_i)$$

$$P(\sigma_i) = \frac{e^{\beta h_i^{loc} \sigma_i}}{2 \operatorname{ch} \beta h_i^{loc}}$$

therefore the correlation function

$$C_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle =$$

$$= \begin{cases} i \neq 0 & \langle \sigma_i \rangle \langle \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = 0 \\ i = 0 & 1 - m^2 \end{cases}$$

Only local correlations

Hypotesis of MFT breakdown at the critical point!