On the relation btw thermodynamic limit & SSB (1)

for ∞ dimensional ising model MFT is exact

$J_{ij} = \frac{J}{N} \quad \forall i,j$ long-range interactions

$$Z(T,h) = \int \frac{dm}{\sqrt{2\pi N J \beta}} e^{-\beta \Psi(m)}$$ Exact Solution for FINITE N

$$\Psi(m) = N\left\{ \frac{J}{2} m^2 - \frac{1}{\beta} \ln 2\,ch[\beta(Jm+h)] \right\}$$

therefore the distribution of the local spin ~~average~~

~~$m \times \langle\sigma\rangle$~~

site averaged magnetization

$m = \frac{1}{N} \sum_{i=1}^{N} \sigma_i$

$$P(m) = \frac{e^{-\beta \Psi(m)}}{\int dm\, e^{-\beta \Psi(m)}}$$

m has always normal flucts in the sense that

$$\langle (m - \langle m \rangle)^2 \rangle \sim \frac{1}{N}$$

this can be easily obtained by looking

at the minima of Ψ and expanding around these minima

(2)

$T > T_c$ only one minima $\bar{m}(h)$

$$\bar{m} : \frac{\partial \Psi}{\partial m} = 0 \Rightarrow \bar{m} = tgh[\beta(J\bar{m}+h)]$$

the second derivative is

$$\frac{\partial^2 \Psi}{\partial m^2} = J[1 - \frac{\beta J}{ch^2(\beta(Jm+h))}]N$$

$$1 - \frac{1}{ch^2} = \frac{ch^2-1}{ch^2} = \frac{ch^2-(ch^2-sh^2)}{ch^2} = tgh^2$$

$$\frac{1}{ch^2} = 1 - tgh^2$$

$$\frac{\partial^2 \Psi}{\partial m^2} = JN[1 - \beta J(1-m^2)]$$

here $\beta_c J = 1 \Rightarrow T_c \Rightarrow \beta J = \frac{T_c}{T}$

$$\frac{\partial^2 \Psi}{\partial m^2} = JN\left(1 - \frac{T_c}{T}(1-m^2)\right)$$

as $T > T_c$ $\underset{(h=0}{(\bar{m}=0)}$ $\frac{\partial^2 \Psi}{\partial m^2} > 0$

any way is always > 0

then fluctuations of m are subjects to gaussian distribution

$$P(m) \sim e^{-\frac{1}{2}\left(\frac{\partial^2 \Psi}{\partial m^2}\right)_{\bar{m}}(m-\bar{m})^2}$$

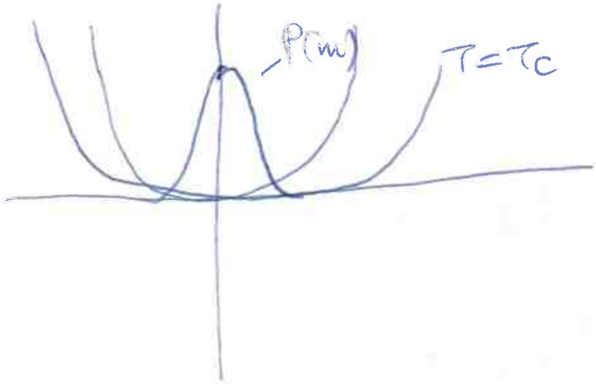
with variance

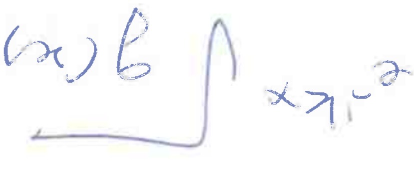
$$\sigma^2 = \frac{1}{\left(\frac{\partial^2 \varphi}{\partial m^2}\right)_{\bar{m}}} = \frac{1}{NJ\left(1 - \frac{T_c}{T}\left(1 - \bar{m}^2(h)\right)\right)} \to 0$$

but for $h = 0$ and $T \to T_c^+$ $\bar{m} = 0$

$$\sigma^2 = \frac{1}{NJ\left(1 - \frac{T_c}{T}\right)} \longrightarrow \infty \text{ for } N \text{ fixed}$$

this simply means that second order term vanishes and you have to take into account further terms



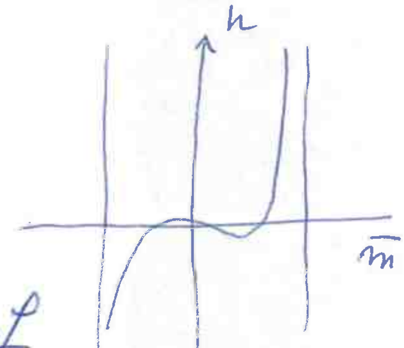


What happens as $T < T_c$

$\rightarrow$ two minima as $h = 0$

given by solution of

$\bar{m} = \tanh[\beta(J\bar{m} + h)]$



Here the MFT $\equiv$ solution of the steepest descent

NON COMMUTATION OF TWO LIMITS

$N \to \infty$ & $h \to 0$

Thermodynamic limit

1) $\lim_{h \to 0^+} \lim_{N \to \infty} \bar{m} = m_+ \quad T < T_c$

2) $\lim_{N \to \infty} \lim_{h \to 0^+} \bar{m} = 0$

SSB occurs only at the thermodyn limit
As 2nd order phase transition

Consider the protocol # 2

$h = 0$

$$\psi(m) = N \left\{ \frac{J}{2} m^2 - \frac{1}{\beta} \ln 2\mathrm{ch}[\beta(Jm)] \right\}$$

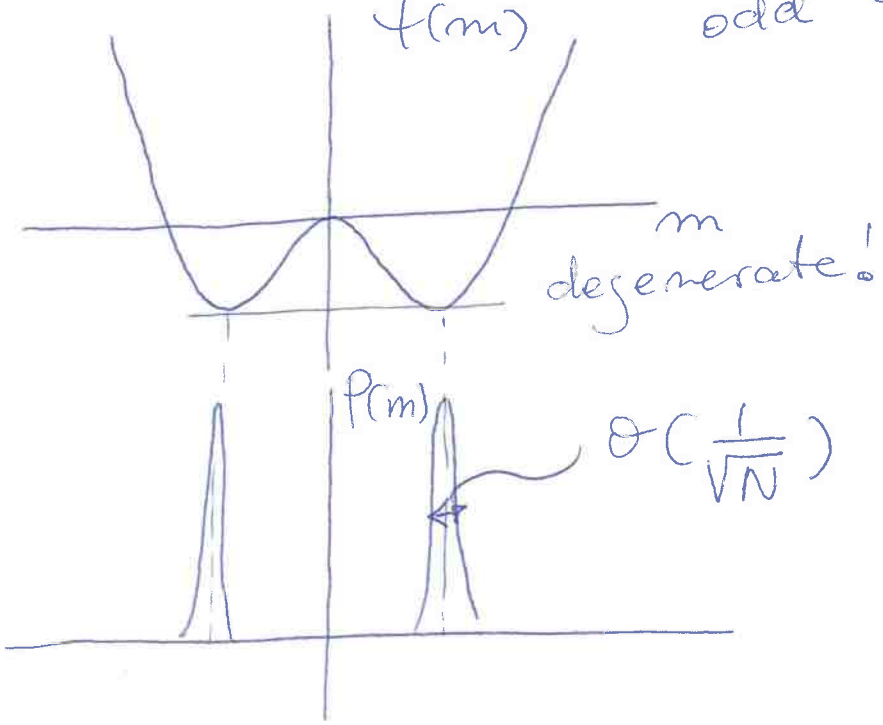
Finite

is SYMMETRIC! $\psi(-m) = \psi(m)$

→ $P(m)$ is SYMMETRIC

$$\left\langle \frac{1}{N} \sum_i \sigma_i \right\rangle = \langle m \rangle = \int dm\, m\, P(m) = 0$$

odd even



No SSB

Consider now the opposite limit
$h \neq 0$ but small

$$ch[\beta(Jm+h)] = ch(\beta Jm) + sh[\beta(Jm)]\beta h + O(h^2)$$

$$\ln 2ch[\beta(Jm+h)] =$$

$$= \ln[2\,ch(\beta Jm)(1 + tgh(\beta Jm)\beta h)] =$$

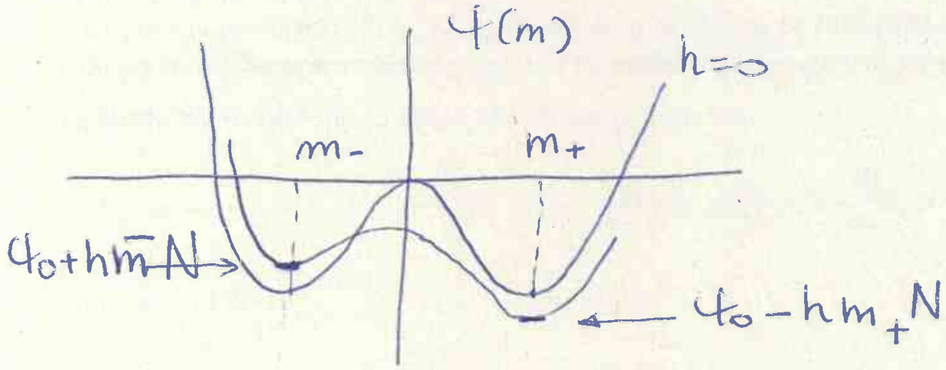
$$= \ln[2\,ch(\beta Jm)] + \beta h\, tgh(\beta Jm)$$

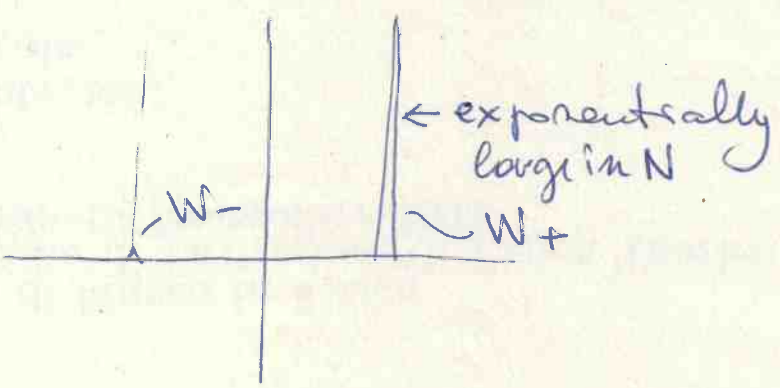
$$-\frac{1}{\beta} \ln 2ch[\beta(Jm+h)] =$$

$$= -\frac{1}{\beta} \ln 2ch[\beta(Jm+h)] - h\, tgh(\beta Jm)$$

$\bar{m}$ around the ~~en~~ minima

$$\psi(m,h) = \psi(m,h=0) - h\bar{m}N$$





$$\Psi(m_+) - \Psi(m_-) = 2 h m_+ N$$

since $m_- = -m_+$

$$\frac{W_+}{W_-} = e^{\beta[\Psi(m_+) - \Psi(m_-)]} = e^{\beta 2 N h m_+}$$

if we take FIRST the limit $N \to \infty$ for finite N

$$\frac{W_+}{W_-} \longrightarrow \infty$$

and $\langle \frac{1}{N} \Sigma_i \sigma_i \rangle = m_+$

even if AFTER we switch $h \to 0$