

Mean-Field Susceptibility and critical exponent ν

①

let's consider non homogeneous field h_i
then

$$m_i = \tanh [\beta (\sum_k J_{ik} m_k + h_i)]$$

$$X_{ij} = \frac{\partial m_i}{\partial h_j}$$

→ real linear response $X_{ij} = \beta C_{ij}$
non local susceptibility
but evaluated in the homogeneous limit

$$X_{ij} = \frac{1}{ch^2(m)} \beta [\sum_k J_{ik} X_{kj} + \delta_{ij}]$$

now consider $h_i = h \forall i \Rightarrow$ homogeneous limit

$$\text{then } ch^2(m) = ch^2[\beta (\sum_k J_{ik} m + h)]$$

and from

$$ch^2 - sh^2 = 1 \quad 1 - \tanh^2 = \frac{1}{ch^2}$$

then

$$X_{ij} = \beta (1 - m^2) [\sum_k J_{ik} X_{kj} + \delta_{ij}]$$

homogeneous limit

as homogeneous limit is concerned

$$X_{ij} = X(R_i - R_j)$$

↑
lattice site

lattice Fourier transform

$$f(\vec{R}) = \frac{1}{N} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{R}} f(\vec{k})$$

$$f(\vec{k}) = \sum_{\vec{R}} e^{-i\vec{k} \cdot \vec{R}} f(\vec{R})$$

$\vec{R}$ is defined on a lattice

both $J_{\vec{k}}$ and $\chi_{\vec{k}}$ depends only on distance

$$\sum_{\vec{k}} J_{\vec{k}} \chi_{\vec{k}} = \sum_{\vec{R}'} \overbrace{J(\vec{R}-\vec{R}') \chi(\vec{R}'-\vec{R}'')}^{\text{convolution}} =$$

$$= \frac{1}{N^2} \sum_{\vec{k}, \vec{k}'} e^{i\vec{k}(\vec{R}-\vec{R}')} e^{i\vec{k}'(\vec{R}'-\vec{R}'')} J(\vec{k}) \chi(\vec{k}') =$$

$$= \frac{1}{N^2} \sum_{\vec{k}, \vec{k}'} e^{i\vec{k}\vec{R}} e^{-i\vec{k}'\vec{R}''} J(\vec{k}) \chi(\vec{k}') \underbrace{\sum_{\vec{R}'} e^{i(\vec{k}'-\vec{k})\vec{R}'}}_{N \delta_{\vec{k}, \vec{k}'}}$$

$$= \frac{1}{N} \sum_{\vec{k}} e^{i\vec{k}(\vec{R}-\vec{R}'')} \underbrace{J(\vec{k}) \chi(\vec{k})}_{\text{Fourier transform of a convolution}}$$

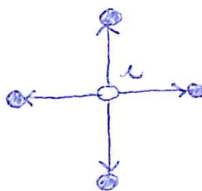
then by Fourier transform of a lattice

$$\chi(\vec{k}) = \beta(1-m^2) [J(\vec{k}) \chi(\vec{k}) + 1]$$

Fourier transform of a Kronecker delta

$$\sum_{\vec{R}} e^{-i\vec{k}\vec{R}} \delta_{\vec{R}} = 1$$

$$J(\vec{R}) = J \sum_{\vec{\delta}} e^{-i\vec{k} \cdot \vec{\delta}}$$



$$J(\vec{r}) = 2J \sum_{\alpha}^d \cos(k_{\alpha} a) \quad (\text{hypercubic lattice}) \quad (3)$$

for small k expand

$$J(\vec{r}) = 2J \sum_{\alpha}^d 1 - \frac{k_{\alpha}^2 a^2}{2} = zJ - Ja^2 |k|^2$$

$$\chi(k) (1 + \beta(1-m^2)J(k)) = \beta(1-m^2)$$

$$\chi(k) = \frac{\beta(1-m^2)}{1 - \beta(1-m^2)J(k)} = \frac{\beta(1-m^2)}{1 - \beta(1-m^2)zJ + \beta(1-m^2)Ja^2|k|^2}$$

$$\chi(k) = \frac{1}{\frac{1 - \beta(1-m^2)zJ}{\beta(1-m^2)} + Ja^2|k|^2}$$

$\chi_0^{-1} \leftarrow \text{thermodynamic susceptibility at } k=0$

$$\chi(\vec{r}) = \frac{\chi_0}{1 + \chi_0 Ja^2|k|^2} = \frac{\chi_0}{1 + \xi^2|k|^2}$$

Ornstein-Zernike type Correlation Function

ξ is the correlation length

$$\xi^2 = \chi_0 Ja^2 \quad \xi \propto |T - T_c|^{-\nu}$$

$$\nu = \frac{1}{2}$$

diverges at critical point!
long-range correlations

but performing the calculations within single site effective Hamiltonian

(4)

$$H_{\text{eff}} = - \sum_i h_i^{\text{loc}} \sigma_i$$

$\Rightarrow$ all sites have independent distribution

$$P(\{\sigma_i\}) = \prod_i P(\sigma_i)$$

$$P(\sigma_i) = \frac{e^{\beta h_i^{\text{loc}} \sigma_i}}{2 \cosh \beta h_i^{\text{loc}}}$$

therefore the correlation function

$$C_{ij} = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle =$$

$$= \begin{cases} \lambda \neq 0 & \langle \sigma_i \rangle \langle \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = 0 \\ \lambda = 0 & 1 - m^2 \end{cases}$$

Only local correlations

Hypothesis of MFT breakdown
at the critical point!