Correlation function in the mean field approximation

In MF Teory correlation functions must be evaluated through response functions.

In the Ising model

$$\chi_{ij} = \frac{\partial \langle \sigma_i \rangle}{\partial h_j} = \beta \left[\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \right]$$

and for translational invariance

$$\chi_{ij} = \chi(\vec{R_i} - \vec{R_j}) \quad \text{with } \vec{R_i} \text{ being a lattice site}$$

by lattice-Fourier transforms

$$f(\vec{R_i}) = \frac{1}{N} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{R_i}} f(\vec{k}) \qquad f(\vec{k}) = \sum_{\vec{R}} e^{-i\vec{k}\cdot\vec{R}} f(\vec{R})$$

with $\vec{k}$ discrete and belonging to 1st Brillouin zone

one has

$$\chi(\vec{k}) = \beta(1-m^2)\left[J(\vec{k})\,\chi(\vec{k}) + 1 \right]$$

$$\chi(\vec{k}) = \frac{\beta(1-m^2)}{1 - \beta(1-m^2)\left[J(k) + J(0) - J(0)\right]}$$

Introducing $\chi(0) = \dfrac{\beta(1-m^2)}{1-(1-m^2)J(0)}$

$$\chi(\vec{k}) = \frac{\beta(1-m^2)}{1-\beta(1-m^2)J(0)-\beta(1-m^2)[J(k)-J(0)]}$$

$$\boxed{\chi(\vec{k}) = \frac{1}{\chi^{-1}(k=0)-[J(\vec{k})-J(0)]}}$$

the Ornstein-Zernike form is obtained in the hydrodynamic limit $k\to 0$

$$\chi(\vec{k}) \simeq \frac{1\ \chi(0)}{\chi(0)(\chi(0)^{-1}+Ja^2|k|^2)}$$

$$\boxed{\chi(\vec{k}) = \frac{\chi(k=0)}{1+\xi_c^2|k|^2}}$$

$$\xi_c^2 = Ja^2\chi(0)$$

as $T\to T_c^+$ $\chi(0)\to\infty$ as $(T-T_c)^{-\gamma}$ $\gamma=1$

and $\xi_c \sim |T-T_c|^{-\nu}$ $\nu = 1/2$

In d=3 we can transform back to real space the obtained estimate using the standard (continuous) Fourier transform

$$\frac{1}{N}\sum_{\vec{k}}(\ldots) \longrightarrow \frac{1}{N}\frac{1}{(\Delta k)^3}\int d^3k\,(\ldots)$$

$$\Delta k = \frac{2\pi}{L}\ (\text{P.B.C.}) \longrightarrow \frac{L^3}{(2\pi)^3 N}\int d^3k\,(\ldots) = \frac{a^3}{(2\pi)^3}\int d^3k\,(\ldots)$$

$$\chi(\vec{R}) \simeq \frac{a^3}{(2\pi)^3}\int d^3k\ \frac{\chi(k=0)\,e^{i\vec{k}\cdot\vec{R}}}{1+(\xi k)^2}$$

the approximation is valid when $|\vec{R}| \gg a$ where a is the lattice spacing ($L = N^{1/3}a$)

$$\chi(\vec{R}) \simeq \frac{a^3}{(2\pi)^3}\,2\pi\int_0^\pi d\theta\int_0^\infty dk\,k^2\sin\theta\ \frac{e^{ikR\cos\theta}\,\chi(0)}{1+(\xi k)^2}$$

where we assume that R is directed along k_z axis

$$\chi(R) \simeq \frac{a^3}{(2\pi)^2}\,2\int_0^\infty dk\,k^2\,\frac{\sin kR}{kR}\ \frac{\chi(0)}{1+(\xi k)^2}$$

$$\left(\int_0^\pi d\theta\sin\theta\,e^{ikR\cos\theta} = \int_{-1}^{1} d\cos\theta\,e^{ikR\cos\theta}\right)$$

$$\chi(R) = \frac{2a^3}{(2\pi)^2}\frac{\chi(0)}{R^3}\underbrace{\int_0^\infty dx\, x^2 \frac{\sin(x)}{x}\frac{1}{1+\left(\frac{\xi}{R}x\right)^2}}_{\frac{\pi e^{-R/\xi_c}}{2\left(\frac{\xi}{R}\right)^2}}$$

this is finite when $\xi \to \infty$

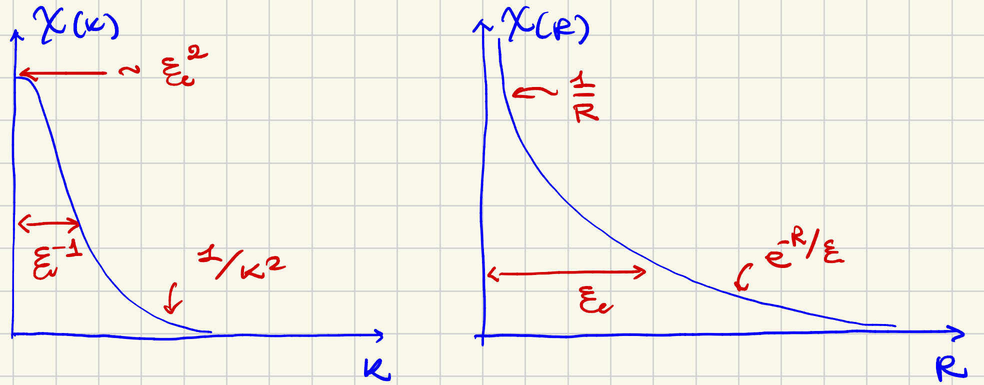
$$\boxed{\chi(R) = \frac{a^3\pi}{(2\pi)^2}\frac{\chi(0)}{\xi_c^2 R}e^{-R/\xi_c} \qquad R \gg a}$$

which clearly shows a decay on a lenght ξ_c and ξ_c is thus the correlation length

Notice the Yukawa form of the response function and the analogy of ξ_c^{-2} with the relativistic m^2 term

$$\chi(k) \simeq \frac{\chi(0)}{1+(\xi_c k)^2} = \frac{\chi(0)\xi_c^{-2}}{\xi_c^{-2}+k^2}$$

this is finite when $\xi \to \infty$



Approaching the critical temperature $\chi(k)$ shows an infrared divergence. For $k < \xi_c^{-1}$ $\chi(k) \sim \xi^2$ and diverges at T_c. For large k the $\chi(k)$ stays finite.

The infrared divergence of $\chi(\bar{R})$ is reflected in the $1/R$ behaviour of $\chi(\bar{R})$ when $\xi_c \to \infty$

Ginsburg Criterion

The validity of the Mean Field Theory is dictated by the smallness of fluctuations in relative terms one can evaluate the scaling of

$$\frac{C(R)}{m^2} = \frac{\langle \sigma(R)\sigma(0)\rangle - \langle \sigma \rangle^2}{\langle \sigma \rangle^2}$$

as $T \to T_c^-$

$C(R)$ is evaluated through $\chi(R) = \beta C(R)$

in $d=3$

$$\chi(R) = C\chi(0)\frac{e^{-R/\xi_c}}{R\,\xi_c^2} \qquad \text{as } R \gg a$$

and therefore $\langle \sigma(R)\sigma(0)\rangle - \langle \sigma(R)\rangle\langle\sigma(0)\rangle \to 0$

as $R \gg \xi_c$ that means $\langle \sigma\sigma\rangle \simeq \langle\sigma\rangle\langle\sigma\rangle$

when $R \gg \xi_c$ but $\xi_c \to \infty$ at the transition!

We can evaluate the scaling of

$$\frac{\chi(R)}{m^2} \quad \text{with } \xi_c$$

~~since~~ as $T \to T_c$ $m \to 0$ and $m^2 \simeq (T-T_c)^{2\beta}$ ($\beta = 1/2$ in MFT) and $\xi_c \sim (T-T_c)^{-\nu}$ ($\nu = 1/2$ in MFT) then in Mean Field Theory $m^2 \sim 1/\xi_c^2$

and

$$\frac{\chi(R)}{m^2} \sim \xi_c^2 \, \chi(0) \frac{e^{-R/\xi_c}}{R\,\xi_c^2}$$

however $\chi(0)$ diverges as $(T-T_c)^{-\gamma}$ ($\gamma = 1$ in MFT) and therefore $\chi(0) \simeq \xi_c^2$ in MFT

$$\frac{\chi(R)}{m^2} \sim \xi_c^4 \frac{e^{-R/\xi_c}}{R\,\xi_c^2} = \xi_c \left[\frac{\xi}{R} e^{-R/\xi} \right] = \xi_c f(R/\xi)$$

in generic d dimensions one has

$$\frac{\chi(R)}{m^2} \propto \xi_c^{4-d} f(R/\xi_c)$$

Indeed generalising the hydrodynamic limit to d dimension one gets

$$\chi(R) \sim \chi(0) \int d^d k \, e^{i\vec{k}\cdot\vec{R}} \frac{1}{1+(\xi k)^2}$$

approximating the relevant region of small k

$e^{i\vec{k}\cdot\vec{R}} \simeq 1$ for $k\xi_c \ll 1$ (infra-red divergence)

$$\chi(R) \sim \chi(0)\Omega_d \underbrace{\int_0^{1/\xi} dk\, k^{d-1} \frac{1}{1+(\xi k)^2}}_{\frac{1}{\xi_c^d} \underbrace{\int_0^1 dx \frac{x^{d-1}}{1+x^2}}_{\text{converges}}}$$

$$\frac{\chi(R)}{m^2} \sim \frac{\overset{\xi_c^2}{\chi(0)}}{\underset{1/\xi_c^2}{m^2}} \frac{1}{\xi^d} \simeq \xi_c^2 \xi_c^2 \frac{1}{\xi^d} = \xi_c^{4-d}$$

therefore fluctuations are negligible when $d \geq 4$ (?) this gives the Ginzburg criterion for validity of MFT for scalar order parameter