Ginsburg Criterion for the validity of the mean-field theory

Mean field theory neglects correlations that is is equivalent to consider an effective Hamiltonian in which the quadratic interaction is linearised

Another way of introducing MFT in Ising is to consider each spin as

$$\sigma_i = \langle \sigma_i \rangle + \delta\sigma_i \qquad (\text{formal } \sigma_i = \pm 1)$$

$$H[\sigma] = -\frac{J}{2} \sum_{\langle ij \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

$$H[\sigma] = H[\langle\sigma\rangle] + \left.\frac{\partial H}{\partial \sigma}\right|_{\langle\sigma\rangle} \delta\sigma + O(\delta\sigma^2)$$

functional derivative

$$H[\sigma] \simeq H[\langle\sigma\rangle] - J \sum_i \delta\sigma_i \sum_j^{(i)} \langle \sigma_j \rangle - h \sum_i \delta\sigma_i$$

the replacing again $\delta\sigma_i = \sigma_i - \langle \sigma_i \rangle$

one gets with $m_i = \langle \sigma_i \rangle$

$$H \simeq -\frac{J}{2}\sum_{\langle ij\rangle} m_i m_j \cancel{- h \sum_i m_i}$$ subtract

$$- J \sum_i (\sigma_i - m_i) \sum_j^{(i)} m_j - h \sum_i (\sigma_i - \cancel{m_i})$$

$$= \frac{J}{2}\sum m_i m_j - \sum_i \sigma_i \Big(\sum_j^{(i)} m_j + h \Big) = H_{eff}$$

and

$$-\frac{1}{\beta}\ln \Psi(m) = \mathrm{tr}_{\{\sigma\}}\, e^{-\beta H_{eff}}$$

with

$$\Psi(m) = \frac{J}{2}\sum_{\langle ij\rangle} m_i m_j - \frac{1}{\beta}\sum_i \ln 2\,\mathrm{ch}\, \beta\Big[J\Big(\sum_j^{(i)} m_j + h\Big)\Big]$$

Landau potential

$$H(\sigma) = -\frac{J}{2}\sum_{\langle ij\rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

$$\Delta\sigma + \langle\sigma\rangle = \sigma$$

$$H = H(\langle\sigma\rangle) + \frac{\partial H}{\partial \sigma}\Delta\sigma + O(\Delta\sigma^2)$$

therefore MFT hold when the correlation function really factorises

$$C(\vec{R}) = \beta^{-1} \chi(\vec{R}) \quad \text{from linear Response}$$

$$\frac{C(\vec{R})}{m^2} \ll 1$$

$$\frac{\langle \sigma_{i+R}\, \sigma_i \rangle}{\langle \sigma_{i+R} \rangle \langle \sigma_i \rangle} \ll 1$$

Ginsburg Criterion ($T > T_c$)

evaluate $C(\vec{R})$ as $\beta^{-1} \chi(\vec{R})$

and

$$\chi(\vec{R}) = \frac{1}{N} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{R}} \chi(\vec{k})$$

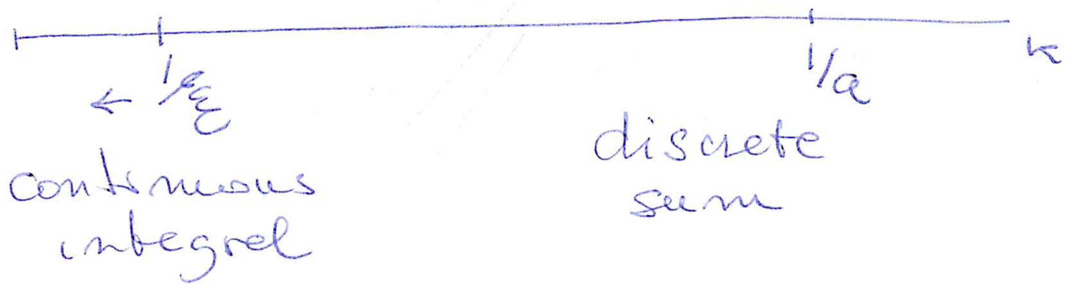
$$= \underbrace{\frac{1}{N} \underset{\Lambda}{\sum_{k}} e^{i\vec{k}\cdot\vec{R}} \chi(\vec{k})}_{\substack{\text{discrete} \\ \text{regular}}} + \underbrace{\frac{1}{N} \overset{\wedge}{\sum_{k}} e^{i\vec{k}\vec{R}} \chi(k)}_{\substack{\text{continuous} \\ \text{divergent at } T_c}}$$

Λ some infrared cutoff

$$\downarrow$$

$$\chi^a$$

what is Λ



assume that $\Lambda \sim 1/\xi_c$ such as

$$\chi(R) = \chi^{d} + \frac{1}{N}\sum_{k \neq 0} e^{i\vec{k}\cdot\vec{R}}\, \chi(\vec{k})$$

in the Ising model at MF level

$$\chi(\vec{k}) = \beta \frac{1-m^2}{1-(1-m^2)\beta J(\vec{k})}$$

with $J(\vec{k}) = z J \sum_{\alpha=1}^{d} \cos(k_\alpha a)$

as $\vec{k} \neq 0$ and $T \to T_c^-$ $m \to 0$ and

$$\chi(\vec{k} \neq 0) \to \beta_c \frac{1}{1-\beta_c J z \sum_\alpha \cos(k_\alpha a)}$$

since $\beta_c J z = 1$ but $\cos(k_\alpha a) < 0$ as $k \neq 0$

the expression do not diverge!

$$\chi^{d} = \frac{L^d}{(2\pi)^d}\frac{1}{N}\int_0^{\Lambda} d^dk\, e^{i\vec{k}\cdot\vec{R}} \frac{\chi_0}{1+(\xi_c k)^2} = \frac{a^d}{(2\pi)^d}\Omega_d \int_0^{\Lambda} dk \frac{k^{d-1}}{1+(\xi_c k)^2}$$

approximate $k \simeq 0$

$$\chi^{dd} \sim \frac{a^d \chi_0}{(2\pi)^d} \Omega_d \underbrace{\int_0^{1/\xi_c} dk \frac{k^{d-1}}{1+(\xi_c^2 k^2)}}_{\frac{1}{\xi_c^d} \underbrace{\int_0^1 dx \frac{x^{d-1}}{1+x^2}}_{\text{finite for any } d>0}}$$

express all quantities as a function of ξ_c

$\chi_0 \sim \xi_c^2$ so that

$\xi_c \sim (T-T_c)^{-\nu}$ $\nu = 1/2$
$\chi_0 \sim (T-T_c)^{-\gamma}$ $\gamma = 1$
$m \sim (T_c-T)^{\beta}$ $\beta = 1/2$
MFT exponents

$$\chi^{dd} \sim \frac{1}{\xi_c^{d-2}} \quad \text{as } \xi_c \to \infty \quad \text{diverges when } d<2$$

Now compare

$$\frac{C(\vec{R})}{m^2} = \frac{\chi(\vec{R})}{\beta m^2} \simeq \frac{\chi^d(R)}{\beta m^2}$$

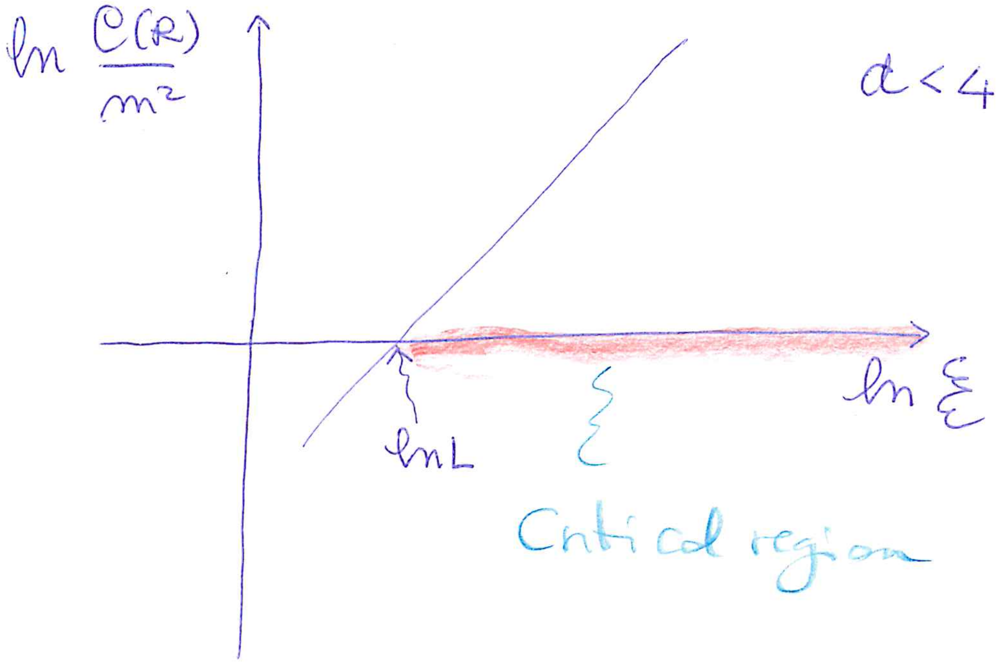
since $m^2 \sim \frac{1}{\xi_c^2}$ we have

$$\underbrace{\frac{C(R)}{m^2}}_{\text{long distance}} \simeq \frac{\xi_c^2}{\xi_c^{d-2}} = \xi_c^{4-d}$$ diverges as $d<4$

for dimensional reason

dimensionless

$$\frac{C(R)}{m^2} = \left(\frac{\xi_c}{L}\right)^{4-d}$$ where L is a non-universal lenght which depends on the system's params



$$\ln \frac{C(R)}{m^2} = (4-d)\ln \xi - (4-d)\ln L$$

as $\xi > L$ $\longrightarrow$ critical region where critical exponents deviates from that of mean field