# Ising model $d=1$ $(h=0)$ equivalence with free fermion gas

$$\mathcal{H} = -J \sum_i \underbrace{\sigma_i \sigma_{i+1}}_{\text{local bond variable } p_i} \qquad (h=0)$$

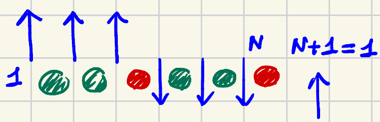
$$p_i = \sigma_i \sigma_{i+1}$$

| $\sigma_i$ | $\sigma_{i+1}$ | $p_i$ |
|---|---|---|
| $\uparrow$ | $\uparrow$ | 1 |
| $\uparrow$ | $\downarrow$ | -1 |
| $\downarrow$ | $\uparrow$ | -1 |
| $\downarrow$ | $\downarrow$ | 1 |

aligned

not aligned

$p_i = -1$ means a defect in the bond $i - i+1$




6 spin + periodic boundary condition

$$p_i = 1 - 2 d_i \nearrow \begin{array}{l} p_1 = 1 \quad d_i = 0 \\ p_i = -1 \quad d_i = 1 \end{array}$$

$d_i$ is the number of defects at bond $i - i+1$

$$\mathcal{H} = -J \sum_i p_i = -J \sum_i (1 - 2 d_i) = -JN + 2J \sum_i d_i$$

$$\mathcal{Z} = \sum_{\{\sigma\}} e^{-\beta H} = e^{\beta J N} \sum_{\{d\}} e^{-\beta 2J \sum_i d_i}$$

$$\mathcal{Z} = e^{\beta J N} \prod_1^N \sum_{d_i=0}^{1} e^{-2\beta J d_i} = e^{\beta J N} \prod_1^N (1+e^{-2\beta J})$$

partition function of free spinless fermions ($\epsilon-\mu=2J$)

$$\langle d \rangle = \frac{1}{e^{\beta 2J}+1}$$

## Magnetization

$$\sigma_{l+1} = \sigma_l \sigma_{l+1} \cdot \sigma_l = P_l \sigma_l \quad \text{since} \quad \sigma_l^2 = 1$$

$$\sigma_{l+1} = P_l P_{l-1} \cdots P_1 \sigma_1$$

$$\sigma_l = \overbrace{P_{l-1} P_{l-2} \cdots P_1}^{l-1 \text{ terms}} \sigma_1$$

$$M = \sum_1^N \langle \sigma_i \rangle = \sum_1^N \langle \prod_1^{i-1} P_k \rangle \sigma_1$$

I left $\sigma_1$ outside the average meaning we fix it and the eventualy average at the end

Now $P_k = 1-2d_k$ are independent random variables therefore

$$\langle \prod_{1}^{i-1}{}_k \, p_k \rangle = \prod_{1}^{i-1}{}_k \langle p_k \rangle = \prod_{1}^{i-1}{}_k \langle 1-2d_k \rangle = \langle 1-2d \rangle^{i-1}$$

$$M = \sigma_1 \sum_{1}^{N}{}_i \langle 1-2d \rangle^{i-1} = \sigma_1 \sum_{0}^{N-1}{}_i \langle 1-2d \rangle^{i}$$

$$\sum_{0}^{N-1}{}_k a^k = \frac{1-a^N}{1-a}$$

$$M = \sigma_1 \frac{1-\langle 1-2d \rangle^N}{1-\langle 1-2d \rangle}$$

$$\langle 1-2d \rangle = 1 - \frac{2}{e^{2\beta J}+1} = \frac{e^{2\beta J}+1-2}{e^{2\beta J}+1} = \frac{e^{\beta J}-e^{-\beta J}}{e^{\beta J}+e^{-\beta J}} = \mathrm{tgh}(\beta J)$$

$$M = \frac{1-\mathrm{tgh}^N(\beta J)}{1-\mathrm{tgh}(\beta J)}$$

Now perform the thermodynamic limit at $(T>0)$

$$\boxed{\lim_{N\to\infty} M = \frac{1}{1-\mathrm{tgh}(\beta J)}}$$

it is non extensive for any $T>0$

therefore

$$m = \langle \sigma_i \rangle = \lim_{N\to\infty} \frac{M}{N} \to 0 \quad (T>0)$$

Now consider $T \to 0$ at finite $N$

as $\beta \to \infty$

$$\operatorname{tgh} \beta J = \frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}} = \frac{1 - e^{-2\beta J}}{1 + e^{-2\beta J}} \simeq (1 - e^{-2\beta J})(1 - e^{-2\beta J}) \simeq 1 - 2e^{-2\beta J}$$

$$M \simeq \frac{1 - (1 - 2e^{-2\beta J})^N}{1 - (1 - 2e^{-2\beta J})} \simeq \frac{1 - (1 - 2Ne^{-2\beta J})}{2e^{-2\beta J}}$$

$$M \simeq N$$

and therefore

$$\lim_{N \to \infty} \lim_{T \to 0} \frac{M}{N} = 1$$ and there is a phase transition at $T = 0$