Physical quantity	Name of index	Formula	Region	Value of index
specific heat	α	$C = A_{>}\epsilon^{-\alpha}$	$h = 0, \epsilon > 0$	0
specific heat	α'	$C = A_{<}(-\epsilon)^{-\alpha}$	$h = 0, \epsilon < 0$	0 (discontinuity)
susceptibility	γ	$\chi = C_{>}\epsilon^{-\gamma}$	$h = 0, \epsilon > 0$	1
susceptibility	γ'	$\chi = C_{<}(-\epsilon)^{-\gamma'}$	$h = 0, \epsilon < 0$	1
magnetization	β	$\langle\sigma\rangle = \pm B(-\epsilon)^{\beta}$	$h = 0, \epsilon = 0$	1/2
magnetization	δ	$\langle\sigma\rangle \sim h^{1/\delta}$	$\epsilon = 0$	3
correlation length	ν	$\xi \sim \epsilon^{-\nu}$	$h = 0, \epsilon > 0$	1/2
correlation length	ν'	$\xi \sim (-\epsilon)^{-\nu'}$	$h = 0, \epsilon < 0$	1/2
correlation function	η	$G(\mathbf{R}) \sim \lvert\mathbf{R}\rvert^{-(d-2)-\eta}$	$h = 0, \epsilon = 0$	0

Table 11.1. Definitions and mean field theory values of critical indices. The word 'discontinuity' indicates that in mean field theory the specific heat has a discontinuity at $T = T_c$.

11.3 Critical Indices

A wide variety of systems can be described by mean field theories. All these theories are different in that they have different physical quantities playing the roles we have given to the magnetic field, or $T - T_c$, or serving as the order parameters. Usually there is some operator which undergoes a discontinuous jump, or labels the different states which arise at the same physical conditions. The average of this operator is called the order parameter. Then there are fields which serve to break a symmetry (as h does in our example) or to move one away from criticality in coupling strength (as does our $T - T_c$ or ϵ). These different problems do share many common features. For example many different mean field theories have the very same dependence of the order parameter on the appropriate fields, except for some multiplicative constants. To have results which we can compare with experiments and other theories, we would like to extract from the mean field theory some numbers which do not depend upon these multiplicative constants. We seek to gain the further advantage that the numbers which we shall use to characterize the theory will not change as one changes dimensional quantities (like temperature). We achieve this goal by noticing that typically physical quantities behave as powers of one another near the critical point. We have already realized this power law behavior when we were talking about scaling laws and said that h behaved as $\langle\sigma\rangle^3$ and ϵ behaved as $\langle\sigma\rangle^2$. The powers in expressions like this are called *critical indices*.

To represent the result we have developed so far is to make a list of the so-called critical indices which describe this mean field behavior. This listing is done in Table 11.1. One index in question describes the behavior of the magnetization as $T_c - T$ goes to zero from above. The result is that the magnetization is proportional to the square root of $T_c - T$. More generally, one might imagine that in a class of models of this type the magnetization

Physical quantity	Name of index	Mean field theory	Index value $d = 3$	Index value $d = 2$
specific heat	α	0 (discontinuity)	0.104 ± 0.003	0 (log singularity)
susceptibility	γ	1	1.2385 ± 0.0015	7/4
susceptibility	γ'	1	1.23	7/4
magnetization	β	1/2	0.325	1/8
magnetization	δ	3	5.2 ± 0.15	15
correlation length	ν	1/2	$0.632 \pm 0.001 \pm 0.025$	1
correlation functions	η	0	0.039 ± 0.004	1/8

Table 11.2. Data for critical indices. Some of the data were taken from C. Domb, The Critical Point: p. 354. Other values come from Kadanoff *et al.*, op. cit.

There are three sources of data in the Table 11.2. Most of the two-dimensional results are obtained from Onsager's exact solution of the two-dimensional Ising model.[13] The implications of this exact solution were worked up by other people including Yang[14] and myself.[15] Experiment provides an excellent source of critical index data for three-dimensional problems. Later developments, based upon universality concepts, would show that one could equally well use data from ferromagnetic transitions, liquid-gas phase transitions, and transitions within critical mixtures (i.e. ones in which different fluids began to separate from one another), as information about the very same type of phase transition. If we use this idea we can include all of these experimental systems in constructing a table like (11.2). (To make the analysis we must identify the order parameters in each of the experimental transitions. For the liquid-gas phase transition, the order parameter is the density. For the fluid mixing transition, the order parameter is the concentration of one of the constituents.)

The final source of data for this table comes from the analysis of power series for thermodynamic quantities. Books[16] have been written about the generation and analysis of power series. Here I can just give the basic idea. Consider a quantity like the susceptibility in the (say) three-dimensional Ising model. Imagine that we can construct the first 20 coefficients of an expansion of logarithm of this susceptibility in a power series in the coupling K, the series being of the form

$$\ln \chi = \sum_{n=0}^{\infty} A_n K^n . \tag{11.44}$$

[13] L. Onsager's solution is described, for example, in index K. Huang, *Statistical Mechanics* (Wiley (1987)) second edition Chapter 15 and also in C. Itzykson and J.-M. Drouffe, *Statistical Field Theory*, Volume I, Sec. 2.2 (Cambridge University Press, Cambridge 1992).

[14] C. N. Yang, Phys. Rev. 85, 808 (1952).

[15] LPK ESSAYS: Leo Kadanoff, Nuovo Cimento 44B, 296 (1966).

[16] DGL: Volume 3 is devoted to methods of series expansion.