Gaussian Model

The gaussian model for a scalar order parameter

Landau free energy was obtained as extremal of a functional integral describing the partition function

$$\mathcal{Z} = \int \mathcal{D}m(x)\, e^{-\beta\Phi[m]}$$

Landau

$$\sim e^{-\beta\Phi[\bar{m}]}$$

then the free energy is Φ evaluated at the extrema

$$\frac{\delta\Phi}{\delta m(x)} = 0 \Rightarrow m(x) = \bar{m}$$

which is analogous to Curie-Weiss equation for long wave length modes

It is natural to perform an evaluation of fluctuation around the "classical" path $\bar{m}(x)$

Vanishes at $m=\bar{m}$

$$\Psi[m] = \Psi[\bar{m}] + \cancel{\int d^dx \frac{\delta\Psi}{\delta m(x)}} \delta m(x) + \underbrace{\frac{1}{2}\int d^dx \int d^dy \frac{\delta^2\Psi}{\delta m(x)\delta m(y)} \delta m(x)\delta m(y)}_{\text{gaussian term } \Psi_G[\delta m]}$$

as a consequence

$$Z \simeq e^{-\beta\Psi[\bar{m}]} \underbrace{\int \mathcal{D}\delta m(x)\, e^{-\beta\Psi_G[\delta m]}}_{Z_G}$$

$$F = \Psi[\bar{m}] + F_G$$

We can prove that the gaussian term is local i.e. does not involve two integrals by evaluating explicitly the variation of $\Psi[m]$ to second order

$$\Psi[m] = \int d^dx \{ a\, m(x) + b\, m^4(x) + c\,|\nabla m(x)|^2 - h(x)\, m(x) \}$$

let $m(x) = \bar{m} + \delta m$

$$a\bar{m}^2 + \cancel{2a\bar{m}\,\delta m} + a\,\delta m^2$$

$$b\bar{m}^4 + \cancel{4b\bar{m}^3\delta m} + 6b\bar{m}^2\delta m^2$$

$$c|\nabla\bar{m}|^2 + \cancel{2c\nabla\bar{m}\cdot\nabla\delta m} + |\nabla\delta m|^2$$

$$-h\bar{m}\cancel{-h\,\delta m}$$

$\times$ linear terms vanishes around $\bar{m}$ due to $\delta\Psi/\delta m|_{\bar{m}} = 0$

$\times$ zeroth order terms contribute to $\Psi[\bar{m}]$

Gaussian is thus defined by two different functionals

$T > T_c$ $\bar{m} = 0$

$$\Psi_G[\delta m] = \int d^dx \{ a\,\delta m^2(x) + c\,|\nabla\delta m(x)|^2 \}$$

$T < T_c$ $\bar{m} \neq 0$

$$\mathcal{H}_G[\delta m] = \int d^d x \left\{ (a + 6b\bar{m}^2)\delta m^2(x) + c|\nabla \delta m(x)|^2 \right\}$$

which have both the same form.

It is useful to express the gaussian model in k space

$$\delta m(x) = \int \frac{d^d k}{(2\pi)^d} e^{i\bar{k}\cdot\bar{x}} \varphi(k)$$

$$\varphi(k) = \int d^d x \, e^{-i\bar{k}\bar{x}} \delta m(x)$$

$$\mathcal{H}_G[\varphi] = \int \frac{d^d k}{(2\pi)^d} \left\{ a + c|\bar{k}|^2 \right\} \varphi(k)\varphi(-k)$$

where a must be understood as $\nearrow a \quad T > T_c$, $\searrow a + 6b\bar{m}^2 \quad T < T_c$

To explicitly evaluate the functional integral containing $\mathcal{H}_G$ it is useful to discretize the k integral as usual

$$k = \frac{n 2\pi}{L} \quad \Delta k = \frac{2\pi}{L} \quad (\Delta k)^d = \frac{(2\pi)^d}{L^d}$$

$$\mathcal{H}_G[\varphi] = \frac{1}{V} \sum_{\vec{k}} (a + c|\bar{k}|^2) \varphi_k \varphi_{-k}$$

using this discretisation the functional integral becomes a multiple integral transforms into a multiple integral

$$Z_G = \int \mathcal{D}\delta m(x)\, e^{-\beta \mathcal{H}_G[\delta m]}$$

$$\downarrow$$

$$Z_G = \int \frac{\prod_k d^2\varphi_k}{\mathcal{N}}\, e^{-\frac{\beta}{V}\sum_k (a + c|\vec{k}|^2)|\varphi_k|^2}$$

where $\mathcal{N}$ is a normalisation factor and φ_k is a complex variable which obeys

$$\varphi_{-k} = \varphi_k^*$$

therefore

$$d^2\varphi_k = dx_k\, dy_k$$

$$x_k = \mathrm{Re}\,\varphi_k \qquad y_k = \mathrm{Im}\,\varphi_k$$

$$Z_G = \int \frac{\prod_k dx_k\, dy_k}{\mathcal{N}}\, e^{-\frac{\beta}{V}\sum_k (a + c|\vec{k}|^2)(x_k^2 + y_k^2)}$$

It is evident that Z_G is a gaussian integral and each x_k & y_k is independent

$$Z_G = \frac{1}{\mathcal{N}} \Pi_k 2\pi\sigma_k^2 \qquad \sigma_k^2 = \frac{2V}{\beta(a+c|\vec{k}|^2)}$$

$$\ln Z_G = -\Sigma_k \ln(a+c|\vec{k}|^2) + \underbrace{\ln \frac{\Pi_k \frac{2V}{\beta}}{\mathcal{N}}}_{\ln Z_G'}$$

it is important to notice that the most important factor around T_c is the first this is the key factor around T_c indeed

$$\sigma_k^2 \sim \frac{1}{a+c|k|^2}$$

diverges as $T \to T_c$ at $k=0$

Gaussian fluctuations are indeed important around the critical point.

It is important to notice that

$$\langle |\varphi_k|^2 \rangle = 2\sigma_k^2$$

and φ_k is the Fourier transform of the order parameter fluctuations around the Landau mean field prediction.

Therefore gaussian model poses a serious issue on the validity of MFT just around the critical point.

Within gaussian model we can show that even critical indexes i.e. qualitative behaviour of thermody-namic quantities are affected by fluctuations.

Let us consider the specific heath

$$C_V = \left(\frac{\partial U}{\partial T}\right)_h$$

We have seen that within MFT

$$C_V \sim \frac{1}{(T-T_c)^\alpha} \qquad \alpha = 0$$

i.e. we expect a jump in the specific heat around the critical point.

Let's calculate U as

$$U = -\frac{\partial}{\partial \beta} \ln Z$$

$$Z = e^{-\beta \Psi[\bar{m}]} Z_G$$

$$\ln Z = -\beta\Psi[\bar m] + \ln Z_G =$$

$$= -\beta\Psi[\bar m] - \sum_k \ln(a + c|k|^2) + \ln Z_G'$$

MFT

fluctuations

Regular at $T \sim T_c$

$$U = U_{MFT} + \frac{\partial}{\partial\beta}\sum_k \ln(a + c|k|^2) - \frac{\partial}{\partial\beta}\ln Z_G'$$

$$a = a'(T - T_c)$$

$$\frac{\partial}{\partial\beta}\sum_k \ln(a + c|k|^2) = -k_B T^2 \sum_k \frac{a'}{a + c|k|^2}$$

$$U = U_{MFT} + U_0 - k_B T_c^2 \sum_k \frac{a'}{a + c|k|^2}$$

$$C_V = C_V^{MFT} + C_V^0 + k_B T_c^2 \sum_k \frac{(a')^2}{(a + c|k|^2)^2}$$

C_V^G

where the derivative has been chosen w.r.t. the relevant parameter $T - T_c$ –

We know that C_V^{MFT} has a discontinuity at T_c while C_V^0 is regular we show that a more important nonanaliticity ...

ty comes from the fluctuation term -

We have seen that within MFT

$$\xi_c^2 = \frac{c}{a + 6b\bar{m}^2}$$

in the gaussian model we assumed

$$a \begin{cases} \to a & T > T_c \\ \to a + 6b\bar{m}^2 & T < T_c \end{cases}$$

there fore

$$C_V^G = k_B T_c^2 \frac{(a')^2}{c^2} \sum_k \frac{1}{(\xi_c^{-2} + |k|^2)^2}$$

the sums shows an infrared divergence while ultraviolet is prevented by the presence of a finite lattice spacing.

Infrared divergence is evident by transferring the sum into integrals

$$C_V^G \propto \int_0^\Lambda dk \frac{k^{d-1}}{(\xi_c^{-2} + k^2)^2}$$

$$C_V^G \propto \xi_c^{4-d} \int_0^{\Lambda \xi_c} dx \frac{x^{d-1}}{(1 + x^2)^2}$$

$- \quad (1+x \;)$

as $d<4$ the integral has no UV divergence while the prefactor ξ^{4-d} leads to a IR divergence as $\xi \to \infty$

from MFT we know that

$$\xi \to \infty \text{ as } \frac{1}{(T-T_c)^{1/2}}$$

Therefore gaussian model predicts that

$$C_V \sim \frac{1}{(T-T_c)^{\alpha}} \qquad \alpha = \frac{4-d}{2}$$

MFT result ($\alpha=0$) is recovered only for $d=4$.

This is in agreement with the Ginsburg criterion that fluctuations becomes important for $d<4$ for scalar order parameter.