Landau functional in the continuum (small q)

Develop for small $\underline{m}$ and $\underline{h}$
taking only linear terms in $\underline{h}$

$$\Psi(\underline{m}) = \frac{1}{2}(J\underline{m})\cdot\underline{m} - \frac{1}{\beta}\,\mathrm{tr}\ln 2\,\mathrm{ch}[\beta(J\underline{m}+\underline{h})]$$

$$\ln 2\,\mathrm{ch}\,x = \frac{x^2}{2} - \frac{x^4}{12}$$

$$\mathrm{tr}\ln 2\,\mathrm{ch}[\beta(J\underline{m}+\underline{h})] =$$

$$= \underbrace{\frac{1}{2}|\beta(J\underline{m}+\underline{h})|^2}_{(a)} - \underbrace{\frac{1}{12}\left(|\beta(J\underline{m}+h|^2\right)^2}_{(b)}$$

linear terms in h comes only from a)

a) $\frac{1}{2}\,2\,\beta^2\,J\underline{m}\cdot\underline{h}$

b) $\cancel{-\frac{1}{12}\beta^4\,3\,\underline{m}^3\cdot\underline{h}}$ $\sim m^3$

$$\mathrm{tr}\ln 2\,\mathrm{ch}[\beta(J\underline{m}+\underline{h})] \simeq$$

$$\simeq \frac{1}{2}\beta^2|J\underline{m}|^2 + \beta^2 J\underline{m}\cdot\underline{h} - \frac{1}{12}\beta^4(|J\underline{m}|^2)^2$$

Now consider the Fourier space and

develop $\tilde{J}(\vec{k})$ for small $\vec{k}$

$$\tilde{J}(\vec{k}) = J_0 - J_1 |\vec{k}|^2$$

$J_0 = zJ \quad J_1 = Ja^2$ for hypercubic lattices

Let's collect the development for $\varphi(\underline{m})$

$$\varphi(\underline{m}) = \frac{1}{2} \underline{\underline{J}\,\underline{m} \cdot \underline{m}} - \frac{1}{2}\beta |\underline{J}\,\underline{m}|^2 - \beta \underline{J}\,\underline{m} \cdot \underline{h} + \frac{\beta^3}{12}(|\underline{J}\,\underline{m}|^2)^2$$

development in $\vec{k}$ in Fourier space is analogous to gradient development in real space

- $\underline{J}\,\underline{m} \cdot \underline{m} = \frac{1}{N} \sum_{\vec{k}} \tilde{J}(\vec{k}) m(\vec{k}) m(-\vec{k})$
- $|\underline{J}\,\underline{m}|^2 = \frac{1}{N} \sum_{\vec{k}} \tilde{J}^2(\vec{k}) m(\vec{k}) m(-\vec{k})$

This gives to small k

- $\underline{J}\,\underline{m} \cdot \underline{m} \simeq \frac{J_0}{N} \sum_k |m(\vec{k})|^2 - \boxed{\frac{J_1}{N} \sum_k |\vec{k}|^2 |m(k)|^2}$
- $|\underline{J}\,\underline{m}|^2 = \frac{1}{N} \sum_k J_0^2 |m(k)|^2 - \boxed{\frac{2}{N} J_0 J_1 \sum_k |k|^2 |m(k)|^2}$
- $\underline{J}\,\underline{m} \cdot \underline{h} = \frac{J_0}{N} \sum_{\vec{k}} m(k) h(-k)$

here we do not consider terms of order
$|\vec{k}|^2 h m$

$$\left(|\partial \underline{m}|^2\right)^2 \simeq \left(\frac{1}{N}\Sigma_k J_0^2 |m(k)|^2\right)^2$$

here we do not consider terms of
the order
$|\vec{k}|^2 m^4$

Therefore terms proportional to $|\vec{k}|^2$
comes only from THEESE terms
As a consequence

$$\psi(\underline{m}) \cong \frac{a}{N} \Sigma_{\vec{k}} |m(\vec{k})|^2 +$$
$$+ b\left(\frac{1}{N}\Sigma_k |m(\vec{k})|^2\right)^2 +$$
$$+ c\ \frac{1}{N} \Sigma_{\vec{k}} |\vec{k}|^2 |m(\vec{k})|^2$$
$$- \frac{1}{N} \Sigma_{\vec{k}} m(\vec{k})\, h(-\vec{k})$$

Which translates in real space

$$\Psi[m] = \int \frac{d^d x}{a^d}\; a\, m^2(x) + b\, m^4(x) + c\, |\nabla m(x)|^2$$
$$- \int \frac{d^d x}{a^d}\, m(x)\, h(x)$$

to obtain the ~ term we have implicitly assumed $\beta J_0 \simeq 1$ in the vicinity of the critical point.

Coefficients a, b, c are given by

$$\boxed{a = \frac{J_0}{2}(1 - \beta J_0) = J_0 \frac{(T - T_c)}{2T} \simeq \frac{T - T_c}{2}}$$

since $T_c = J_0$

$$c = -\frac{J_1}{2} + \beta J_0 J_1$$

when $\beta J_0 \simeq 1$

$$\boxed{c \simeq \frac{J_1}{2} > 0}$$

$$\boxed{b = \frac{\beta^3}{12} J_0^4 \simeq \frac{J_0}{2} > 0}$$

Dimensions

$[\varphi] = \text{Energy}$

$[J_0] = \text{energy}$

$[T] = \text{energy} \quad (k_B = 1)$

$[J_1] = \text{Energy} \times L^2$

$[a] = [b] = \text{Energy}$

$[c] = \text{Energy} \times L^2$

$[m]$ = dimensionless in both x & k

$[h] = \text{Energy}$