Landau functional for the Ising model

Ising Model finite dimensions

$$\mathcal{H} = -\frac{1}{2}\sum_{\langle ij\rangle} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i$$

general coupling

local fields

$$Z = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{\{\sigma\}} e^{-\beta H_\square} e^{-\beta H_L}$$

Linearize trough a Stratonovich-Hubbard (gaussian) transformation

$$\boxed{e^{-\beta H_\square} = \int \left[\Pi_i \frac{d\varphi_i}{\sqrt{2\pi}}\right] \frac{1}{(\det J)^{1/2}} e^{-\frac{\beta}{2}\underline{\varphi}[J]^{-1}\underline{\varphi} + \beta\underline{\sigma}\cdot\underline{\varphi}}}$$

this works if all eigenvalues of J are strict positive.

This is NOT the case of nearest neighbor interaction on the hyper cubic lattice.

In this case lattice-Fourier transform

diagonalizes $\mathbb{J}$

$$J_{ij} = J(\underbrace{\vec{R}_i - \vec{R}_j}_{\text{a lattice vector}})$$

$$f(\vec{R}) = \sum_{\vec{k}} \frac{e^{i\vec{k}\cdot\vec{R}}}{N}\,\tilde{f}(\vec{k})$$

$$\tilde{f}(\vec{k}) = \sum_{R} e^{-i\vec{k}\cdot\vec{R}} f(\vec{R})$$

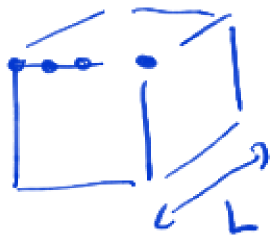
Periodic Boundary Conditions

$$\boxed{\vec{k} = \frac{2\pi}{L}\vec{m}}$$

$\vec{m}$ index vector of natural numbers components

$m_k = 0, \pm 1, \pm 2 \ldots \pm \frac{N}{2}$

for a cubic lattice $L = Na$



as $L \to \infty$

$$f(\vec{R}) \simeq \frac{a^d}{(2\pi)^d}\int d^d\vec{k}\, e^{i\vec{k}\cdot\vec{R}}\,\tilde{f}(\vec{k})$$

$$\tilde{f}(\vec{k}) \simeq \frac{1}{a^d}\int d^d\vec{R}\, e^{-i\vec{k}\cdot\vec{R}} f(\vec{R})$$

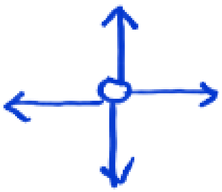
$$\frac{1}{2}\underline{\varphi}\,\mathbb{J}\,\underline{\varphi} = \frac{1}{2}\sum_{ij}\varphi_i J_{ij}\varphi_j =$$

$$= \frac{1}{2} \sum_{\vec{R}\vec{R}'} \varphi(\vec{R}) J(\vec{R}-\vec{R}') \varphi(\vec{R}')$$

$$= \frac{1}{2} \sum_{\vec{k}} \tilde{\varphi}(\vec{k}) \tilde{J}(\vec{k}) \tilde{\varphi}(-\vec{k})$$

$$\tilde{J}(\vec{k}) = \sum_{\vec{R}} e^{-i\vec{k}\cdot\vec{R}} J(\vec{R})$$

for n.n. interaction $J(\vec{R}) = \begin{cases} J & \vec{R} = \vec{\delta} \\ 0 & \text{elsewhere} \end{cases}$



$d=2$ 4 possible $\vec{\delta}$'s

therefore

$$\tilde{J}(\vec{k}) = 2J \sum_{\alpha=1}^{d} \cos(k_\alpha a)$$

where each direction α gives

$-\pi < k_\alpha a < \pi$

and consequently eigenvalues of $\mathbb{J}$ could be negative!!

To circumvent this problem we can add a trivial constant to J redefining

$$\mathbb{J} \longrightarrow \mathbb{J}' = \mathbb{J} + 2zJ\mathbb{1}$$

Now

$$e^{-\beta H_{\square}} = e^{-\frac{\beta}{2}\underline{\sigma}\,\mathbb{J}\,\underline{\sigma}} = e^{-\frac{\beta}{2}\underline{\sigma}(\mathbb{J}'-2zJ\mathbb{1})\underline{\sigma}}$$

$$= e^{-\frac{\beta}{2}\underline{\sigma}\mathbb{J}'\underline{\sigma}}\; e^{\beta zJN}$$

since $\underline{\sigma}\,\mathbb{1}\,\underline{\sigma} = \sum_i \sigma_i^2 = N$

and therefore

$$\operatorname{tr} \underbrace{e^{-\beta H}}_{\text{using } \mathbb{J}} = e^{\beta zJN} \operatorname{tr} \underbrace{e^{-\beta H'}}_{\text{using } \mathbb{J}'}$$

$$F = F' - zJN$$

$$Z = \operatorname{tr} e^{-\beta H} = \int \left(\prod_i \frac{d\varphi_i}{\sqrt{2\pi}}\right) \frac{1}{(\det \mathbb{J})^{1/2}}\; e^{-\frac{\beta}{2}\underline{\varphi}(\mathbb{J}^{-1})\underline{\varphi}} \times$$

$$\times \underbrace{\prod_i 2\operatorname{ch}(\beta(\varphi_i + h_i))}_{\operatorname{tr} e^{\beta \underline{\sigma}(\underline{\varphi}+\underline{h})}}$$

Introducing continuous magnetization

$$\underline{\varphi} = \mathbb{J}\cdot\underline{m}$$

$$\frac{1}{2}\,\underline{\varphi}\,[\mathbb{J}^{-1}]\,\underline{\varphi} = \frac{1}{2}(\mathbb{J}\underline{m})\cdot\underline{m}$$

$$\boxed{Z = \int \Big(\prod_i \frac{dm_i}{\sqrt{2\pi}}\Big)(\det \mathbb{J})^{1/2}\, e^{-\beta\Phi(\underline{m})}}$$

$$\boxed{\Phi(\underline{m}) = \frac{1}{2}(\mathbb{J}\cdot\underline{m})\cdot\underline{m} - \frac{1}{\beta}\,\mathrm{tr}\,\ln 2\,\mathrm{ch}[\beta(\mathbb{J}\underline{m}+\underline{h})]}$$

defining $\mathrm{tr}\,\underline{V} = \sum_i V_i$

where the physical significance of $\underline{m}$ is given by the relation

$$\langle \sigma_i \rangle = \langle m_i \rangle$$

which can indeed be proven for any correlation function of σ

$$\langle \sigma_i \sigma_j \ldots \sigma_k \ldots \rangle = \langle m_i m_j \ldots m_k \ldots \rangle$$