Landau theory as extrema of Landau action

for a discrete set of lattice variable we wrote in the Ising model

$$\mathcal{Z} = \int \left(\Pi_i \frac{dm_i}{\sqrt{2\pi}}\right) (\det J)^{1/2} e^{-\beta \Psi(\underline{m})}$$

then we have linearised the Landau "action" $\Psi(\underline{m})$ at small k vector

$$\Psi(\underline{m}) \cong \frac{a}{N} \Sigma_{\vec{k}} |m(\vec{k})|^2 +$$

$$+ b \left(\frac{1}{N} \Sigma_k |m(\vec{k})|^2\right)^2 +$$

$$+ c \; \frac{1}{N} \Sigma_{\vec{k}} |\vec{k}|^2 |m(\vec{k})|^2$$

$$- \frac{1}{N} \Sigma_{\vec{k}} m(\vec{k}) h(-\vec{k})$$

and write the result in the continuum

$$\Psi[m] = \int \frac{d^d x}{a^d} \; a m^2(x) + b m^4(x) + c |\nabla m(x)|^2$$

$$- \underbrace{\int \frac{d^d x}{a^d} m(x) h(x)}$$

in this limit we consider distances $|\vec{r}| \gg a$ but this implies that m instead of depending on a discrete index depends on a continuum variable x (d-dimensional)

Z becomes a functional integral

$$Z = \int \mathcal{D} m(x) \, e^{-\beta \Psi[m]}$$

measure of funct. integral

The "classical" Landau theory for 2^{nd} order phase transition results as the functional steepest descent evaluation of Z

$$Z_{Landau} \simeq e^{-\beta \Psi[\bar{m}]}$$

where $\Psi[\bar{m}]$ is obtained at a given functional value for

$$m(\underline{x}) = \bar{m}(\underline{x})$$

defined by

$$\left.\frac{\delta \Phi[m]}{\delta m(x)}\right|_{m(x)=\bar{m}(x)} = 0$$

Functional Derivatives

a functional is an application which maps a functional space onto a number (real, complex...)
e.g.

$$\mathcal{J} = \int_{-\infty}^{+\infty} dx\, f(x)$$

is a linear functional which maps the set of functions which are integrable in $(-\infty, +\infty)$ to a number $\mathcal{J}$.

Changing the function $f(x)$ will produce a change in $\mathcal{J}$ so we say

$$\mathcal{J}[f]$$

a variation of f will produce a

a variation of f will produce a variation of $\mathcal{J}$

$$\mathcal{J}[f+\delta f] = \int_{-\infty}^{+\infty} dx \, (f(x) + \delta f(x))$$

$$= \mathcal{J}[f] + \delta \mathcal{J}$$

$$\delta \mathcal{J} = \int dx \, \delta f(x)$$

for a more general functional we can write

$$\delta \mathcal{J} = \int dx \, \frac{\delta \mathcal{J}}{\delta f(x)} \delta f(x)$$

Consider the analogy btw this expression and that of a function of N variables

$$f(x_1, x_2 \ldots x_N)$$

$$df = \Sigma_i \frac{\partial f}{\partial x_i} dx_i$$

here x_i becomes an ∞ dimensional function $f(x)$ and the sum transform into an integral.

Let's change the functional

$$J[f] = \int dx\, f^2(x)$$

$$J[f+\delta f] = \int dx\, f^2(x) + 2 f(x)\delta f(x) + O(\delta f^2)$$

here we have

$$\frac{\delta J}{\delta f(x)} = 2 f(x)$$

Now let's consider a more complex example

$$J[f] = \int dx \left(\frac{df}{dx}\right)^2$$

$$J[f+\delta f] = \int dx \left[\frac{d}{dx}\left(f(x)+\delta f(x)\right)\right]^2$$

$$= \int dx \left\{ \left(\frac{df}{dx}\right)^2 + 2\frac{df}{dx}\frac{d\delta f}{dx} + \cancel{\left(\frac{d\delta f}{dx}\right)^2} \right\}$$

$$\delta J = \int dx\, 2 \frac{df}{dx} \frac{d\delta f}{}$$

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inorder to get an expression containing $\delta f(x)$ instead of $\frac{d\,\delta f}{dx}$ we integrate by parts

$$\delta J = 2\int_{-\infty}^{+\infty} d\delta f\ \frac{df}{dx} = 2\,\delta f \frac{df}{dx}\Big|_{-\infty}^{+\infty} - 2\int_{-\infty}^{+\infty} dx\, \frac{d^2 f}{dx^2}\,\delta f$$

assuming integrability of δf the finite part vanishes therefore

$$\frac{\delta J}{\delta f(x)} = -2\frac{d^2 f}{dx^2}$$

Now consider

$$\Psi[m] = \int \frac{d^d x}{a^d}\ \underset{(a)}{a m^2(x)} + \underset{(b)}{b m^4(x)} + \underset{(c)}{c|\nabla m(x)|^2} - \int \frac{d^d x}{a^d}\ \underset{(d)}{h(x) m(x)}$$

$$\frac{\delta \Psi}{\delta m(x)} = \frac{\delta(a)}{\delta m(x)} + \frac{\delta(b)}{\delta m(x)} + \frac{\delta(c)}{\delta m(x)} + \frac{\delta(d)}{\delta m(x)}$$

$$\frac{\delta(a)}{\delta m(x)} = 2a\, m(x)$$

$$\frac{\delta(b)}{\delta m(x)} = 4b\, m^3(x)$$

$$\frac{\delta(c)}{\delta m(x)} = -2c\, \Delta_2 m(x)$$

$$\frac{\delta(d)}{\delta m(x)} = -h(x)$$

The extremal condition is thus

$$\boxed{2a\, m(x) + 4b\, m^3(x) - 2c\, \Delta_2 m(x) = h(x)}$$

the solution of this equation give the function $m(\underline{x})$ in d-dimension which is the non-homogeneous order parameter in the long-wavelength limit.

This equation play the role of the Curie-Weiss equation obtained in the Ising model

It is clear that as $h(x)=0$ a solution $m(\underline{x})=0 \ \forall \underline{x}$ is possible

It is also clear that an homogeneous solution

$$m(\underline{x}) = m \ \forall \underline{x}$$

is possible if and only if

$$h(\underline{x}) = h \ \forall \underline{x}$$

$$2am + 4bm^3 = h$$

which gives $m=0$ when $h=0$
or $m = \pm\sqrt{-\frac{a}{2b}}$ (then $a<0$)

We can get the susceptibility from the previous equation by performing

$$\frac{\delta m(x)}{\delta h(y)} = \chi(x,y)$$

$$2a\chi(x,y) + 12b\, m^2(x)\, \chi(x,y) - 2c\Delta_2 \chi(x,y)$$

$= \delta(x-y)$

when $h(x) \to h \to 0$

translational invariance is restored and $\chi(x,y) = \chi(x-y)$

We can then proceed to Fourier transform the previous equation obtaining

$$2a\,\chi(\vec{k}) + 12\,b\,m^2\chi(\vec{k}) + 2c|\vec{k}|^2\chi(k) = 1$$

since when $h(x) \to h$ $\quad m(x) = m$

$$\chi(\vec{k}) = \frac{1}{\underbrace{[2a + 12bm^2]}_{\chi(0)^{-1}} + 2c|k|^2}$$

$$\boxed{\chi(\vec{k}) = \frac{\chi(0)}{1 + \xi_e^2|\vec{k}|^2}}$$

Ornstein Zernike

$$\chi(0) = \frac{1}{\;}$$

$2a+12bm^2$

$$\boxed{\xi^2 = 2c\chi(0)}$$

as $T\to T_c^+$ $m=0$ $a=a'(T-T_c)$

$$\chi(0) \simeq \frac{1}{2a'(T-T_c)} \sim (T-T_c)^{-\gamma} \quad \gamma^+=1$$

$$\xi^2 \sim (T-T_c)^{-2\nu} \quad \nu = 1/2$$

as $T\to T_c^-$ $m=\sqrt{\frac{a'}{2b}}\,(T_c-T)^{1/2}$ $\beta = 1/2$

$$\chi(0) = \frac{1}{(2a' + \frac{6a'}{b}b)(T_c-T)} = \frac{1}{8a'(T_c-T)} \quad \gamma^-=1$$

coefficient for divergence of χ changes but classical critical exponent is the same