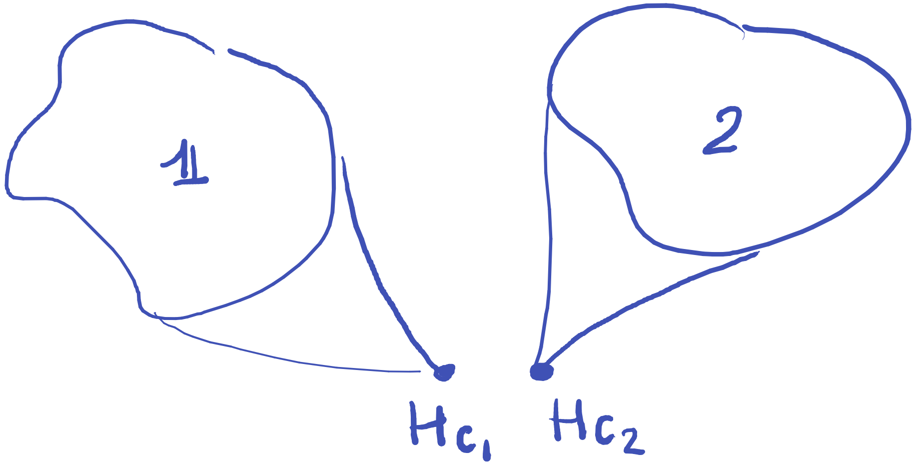
Renormalization Group



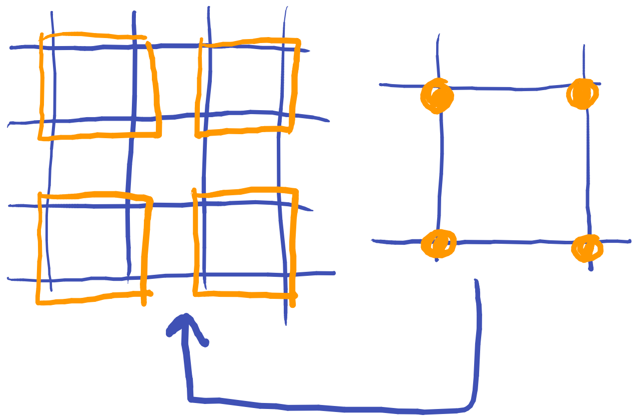
mapping of Hamiltonians onto critical H_c such as

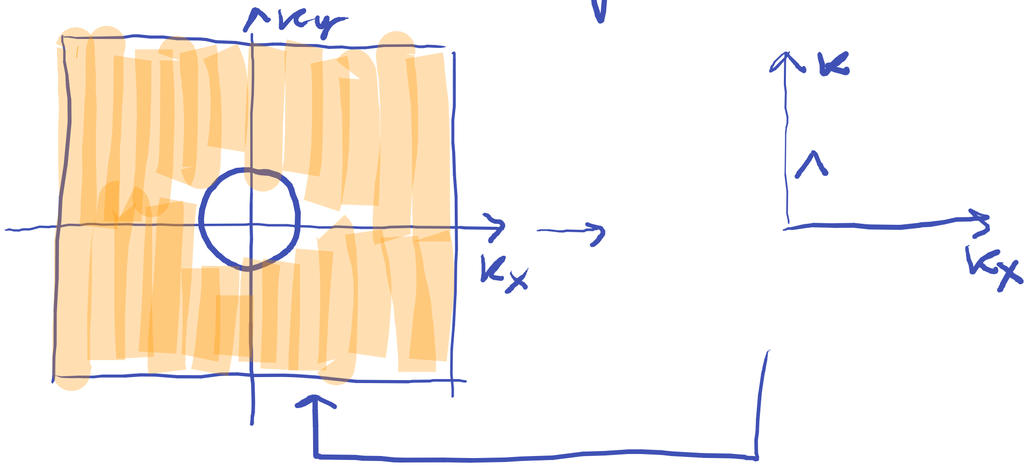
$\mathcal{R} H_{c_1} = H_{c_1}$ is a fix point

Different flavors

Real Space : Kadanoff

Fourier Space : Wilson





Real Spa block variables
integrate (decimate) spins

Integrate out $|\vec{k}|$
modes for continuous H

Knowledge of explicit form of RG transformation
→ calculation of critical indexes

e.g. for ISING model

$$\beta H = -\beta J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \beta h \sum_i \sigma_i$$

$J \xleftarrow{\text{redefine}} \quad \xrightarrow{} h$

$$\bar{J} = f_1(J, h)$$
$$\bar{h} = f_2(J, h)$$

a fixed point is reached when $\begin{cases} \bar{J} = J \\ \bar{h} = h \end{cases}$

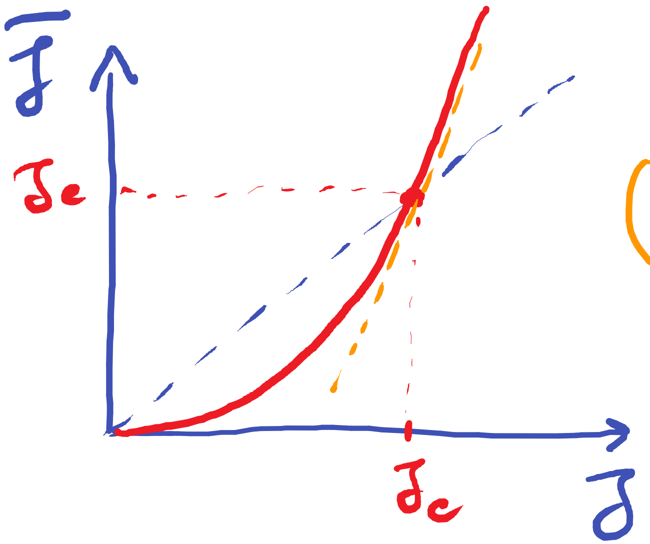
as critical point is reached as $h=0$ let's consider J only

expanding $f_1 = f$ around the critical $J_c = f(J_c)$

$$\bar{J} = \underbrace{f(J_c)}_{J_c} + \left(\frac{df}{dJ}\right)_{J_c} (J - J_c)$$

$$\frac{\bar{J} - J_c}{J - J_c} = \left(\frac{df}{dJ}\right)_{J_c} \simeq \frac{\bar{t}}{t} = L^{x_t}$$

$$\nu = \frac{1}{x_t} \Rightarrow \boxed{\nu = \frac{\ln L}{\ln\left(\frac{df}{dJ}\right)_{J_c}}}$$



$\left(\frac{df}{dJ}\right)_{J_c} > 1 \Rightarrow \nu > 0$ CRITICAL BEHAVIOUR

for a general Hamiltonian which depends on $\{p_i\}$ parameters

$$\bar{p}_i = p_i^c + \sum_j \left(\frac{\partial f_i}{\partial p_j}\right)_{p_c} (p_j - p_j^c) \Rightarrow \text{diagonalise } \left(\frac{\partial f_i}{\partial p_j}\right) \to \lambda_\alpha$$

define new q_i

$$\bar{q}_\alpha = \lambda_\alpha q_\alpha \qquad \boxed{x_\alpha = \frac{\ln L}{\ln \lambda_\alpha}}$$

$x_\alpha > 0$ RELEVANT

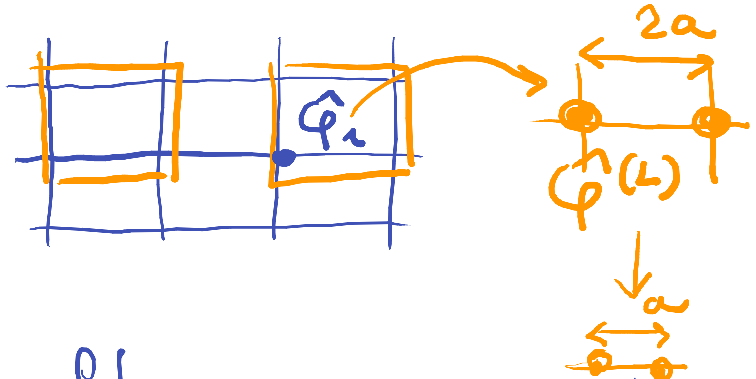
$x_\alpha < 0$ IRRELEVANT

Block variables (real space)

consider a more general lattice Hamiltonian (space is discrete but "spins" are continuous) such that

$$P(\{\hat{\varphi}_i\}) = \frac{1}{Z} e^{-H(\{\hat{\varphi}_i\})}$$ (β absorbed in H)

a block variable on scale L is defined as



$$\hat{\varphi}_J^{(L)} = \frac{1}{L^d} \sum_i^{(J)} \hat{\varphi}_i$$

sum over sites (i) belonging to block J

after summing the block variables one should rescale the lattice of the block spins

$R_i \to \frac{R_i}{L}$ and also the block variables need Rescaling

$$\hat{\varphi}_i \longrightarrow \hat{\varphi}_J^{(L)} \longrightarrow \hat{\varphi}(R_J/L)$$

$$\boxed{\hat{\varphi}(R_J/L) = L^{x_\varphi}\, \hat{\varphi}_J^{(L)}}$$

this rescaling is necessary to obtain non-trivial distribution for the rescaled $\hat{\varphi}(R_J/L)$

if the original distribution for the variables $\hat{\varphi}_i$ is

$$P(\{\hat{\varphi}_i\}) = \frac{e^{-H(\{\hat{\varphi}_i\})}}{Z}$$

then

$$P_L(\{\hat{\varphi}\}) = \int \prod_i d\hat{\varphi}_i \, P(\{\hat{\varphi}_i\}) \prod_J \delta\left(\hat{\varphi}\left(\frac{R_J}{L}\right) - \frac{L^{x_\varphi}}{L^d} \sum_i^{(J)} \hat{\varphi}_i\right)$$

We can see this transformation as transformation on Hamiltonian

$$P(\{\hat{\varphi}_i\}) = \frac{e^{-H(\{\hat{\varphi}_i\})}}{Z} \longrightarrow P_L(\{\hat{\varphi}_j\}) = \frac{e^{-H_L(\{\varphi_j\})}}{Z_L}$$

$$H \longrightarrow H_L = F_L - \ln P_L(\{\hat{\varphi}_j\}) = \mathbb{R}_L H$$

RG transform

and $F_L = -\ln Z_L$

RG acting on P can be expressed by

$$P_L = \mathbb{K}_L P$$

and K_L is an integral operators (integration of block spins)

RG is a Group

$\mathbb{K}_L$ is an integral operator such that $K_{LL'} = K_L K_{L'}$

$$R_{LL'} H = F_{LL'} - \ln K_L K_{L'} P = F_L[H_{L'}] - \ln K_L P_{L'}$$

where $F_L[H_{L'}] = -\ln \operatorname{tr}_{(L)} e^{-H_{L'}}$

$R_{LL'} H = R_L H_{L'}$ therefore $\boxed{R_L(R_{L'}(H)) = R_L R_{L'} H}$

which is a group property.

We Require that the partition function is invariant thus

$$\boxed{F(R_L H) = F(H)}$$

$\boxed{\text{Fixed Points}}$

We have seen R_L as a transformation of $P(\{\hat{\varphi}_i\})$

$$P \xrightarrow{R_L} P_L = R_L(P)$$

$$H \xrightarrow{R_L} H_L = R_L(H)$$

then H^* is a fixed point for Hamiltonian flow if

$$\boxed{R_L(H^*) = H^*}$$

If H depends on a set of parameters $\{p_i\}$

$$\{p_i\} \xrightarrow{R_L} \{\bar{p}_i\}$$

at fixed point $\quad \boxed{\bar{p}_i^* = p_i}$

Linearization around the fixed point

assume RG transformation is regular

i.e. for small τ $\quad H+\tau\delta H$ is close to H

$$\boxed{R_L(H+\tau\delta H) = R_L(H) + \tau L_L(H)\,\delta H}$$

linear operator

correspondingly

$$\bar{p}_\alpha = p_\alpha + \sum_J \left.\frac{\partial f_\alpha^{(L)}}{\partial p_J}\right|_{\{p_\alpha\}} \delta p_J \quad \text{where } \delta p_J = \bar{p}_J - p_J$$

representation of L_L

L_L is a scale transformation only if $H=H^*$

in that case eigenvalues of $\frac{\partial f_\alpha^{(L)}}{\partial p_J}$ $\quad \lambda_\alpha^{(L)} = L^{x_\alpha}$

$\boxed{\text{Linearization around the fixed point}}$

$$R_L(H+\tau\delta H) = R_L(H) + \tau L_L(H)\delta H$$

$$R_{L'}(R_L(H+\tau\delta H)) = R_{L'}(\bar{H}) + \tau L_{L'}(\bar{H})\delta\bar{H}$$

but from group properties of RG previous line is

$$R_{L'L}(H+\tau\delta H) = R_{L'L}(H) + \tau L_{L'L}(H)\delta H$$

$$L_{L'L}(H)\delta H = L_{L'}(\bar{H})\delta\bar{H} = L_{L'}(\bar{H})L_L(H)\delta H$$

$$L_{L'L}(H) = L_{L'}(R_L(H))L_L(H)$$

L is a group only at the fixed point $R_L(H^*) = H^*$

$$\boxed{L_{L'L}(H^*) = L_{L'}(H^*)L_L(H^*)}$$

Linearization around the fixed point

Group properties of L at the fixed point

imply

SCALING

for eigenvalues of L

$$\lambda_\alpha(L'L) = \lambda_\alpha(L')\,\lambda_\alpha(L) \Rightarrow \lambda_\alpha(L) \text{ homogeneous in } L$$

$\lambda_\alpha(L) = L^{x_\alpha}$ such that

$$(L'L)^{x_\alpha} = (L')^{x_\alpha}\,L^{x_\alpha}$$

Scaling holds only at the fixed point

Asymptotic Scaling for block variables

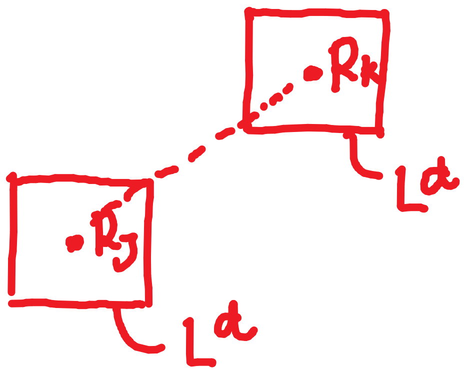
Rescaling of block variables (as L^{x_φ}) ensures the correct asymptotic scaling

$$\hat\varphi_i \longrightarrow \hat\varphi'(R_J/L) = L^{x_\varphi}\hat\varphi_J^{(L)} = L^{x_\varphi}\frac{1}{L^d}\sum_i^{(J)}\hat\varphi_i$$

Correlation function

$$\langle \hat\varphi'(R_J/L)\,\hat\varphi'(R_K/L)\rangle = R_L(C(R_i,R_l))$$

$$C(R_i,R_l) = \langle\hat\varphi_i\,\hat\varphi_l\rangle$$



$$\underbrace{R_L(C)}_{C\left(\frac{|R_J-R_K|}{L}\right)} = L^{2(x_\varphi-d)}\underbrace{\sum_i^{(J)}\sum_l^{(K)}C(R_i,R_l)}_{L^{2d}\,C(r)}$$

$r = |R_J - R_K|$

Asymptotic Scaling for block variables

real space $\quad C\left(\frac{r}{L}, R_L(H)\right) = L^{2x_\varphi} C(r, H)$

Fourier space $\quad C(Lq, R_L(H)) = L^{2x_\varphi - d} C(q, H)$

at the fixed point

$$C(q, H^*) = L^{-2x_\varphi + d} C(Lq, H^*)$$

$Lq = 1$

$$C(q, H^*) = \frac{1}{q^{d-2x_\varphi}} C(1, H^*) \quad \text{const}$$

scaling $\quad x_\varphi = \frac{d+\eta}{2} - 1 \quad$ (cfr x_m order param)

Static correlations

$$\boxed{\lim_{q\to 0} C(q) = \lim_{q\to 0} \frac{1}{q^{2-\eta}}}$$

diverges as $\eta < 2 \Rightarrow x_\varphi < d/2$

regular as $\eta \geqslant 2 \Rightarrow x_\varphi \geqslant d/2$

Trivial fixed points $x_\varphi = \frac{d}{2}$