Solving of Landau functional

The Landau functional represents a free energy which is INVARIANT under a scale transformation

$$F = \int d^d x \{ a m^2(x) + b m^4(x) - c |\nabla m(x)|^2 - h(x) m(x) \}$$

(a) (b) (c) (d)

this should be equal to $\bar{F}$

Under scale transformation

a) $\int d^d x \, a m^2(x) = \int d^d \bar{x} \, \bar{a} \, \bar{m}^2(\bar{x})$

but at the same time the l.h.s.

$$L^d \int d^d \bar{x} \, L^{-x_a} \bar{a} \, L^{-2x_m} \bar{m}^2$$

therefore

$$d - x_a - 2x_m = 0$$

but $x_a = x_t$

$$\boxed{a) \; d - x_t - 2x_m = 0}$$

b) $d - 4x_m - x_b = 0$

c) $d - 2 - 2x_m - x_c = 0$

gradient

d) $d - x_m - x_h = 0$

by choosing $x_t = 1$ in a)

$$x_m = \frac{d-2}{2}$$

substituting in d)

$$x_h = \frac{d+2}{2}$$

These results are dimensionality dependent and gives critical exponent which agrees with MFT only for $d=4$ e.g.

$$\beta = \frac{x_m}{x_t} = \frac{d-2}{4} \qquad (\text{MFT } \beta = \tfrac{1}{2})$$

Landau theory satisfies SCALING only for $d=4$

Scaling of parameters in field theory approach of phase transitions

Consider a FT approach in which

$$Z = \int D\hat{\varphi}\, e^{-H[\hat{\varphi}]}$$

now I introduce $\hat{\varphi}$ as a microscopic variable allowing fluctuations

$\hat{\varphi}(x)$ is a field (can also be vectorial)

$H[\hat{\varphi}]$ may have a general form

Suppose that we can write an expansion which is in both local value of the microscopic order parameter $\hat{\varphi}(x)$ and in its gradients

$$H[\varphi] = \int d^d x \Big\{ \underbrace{a_2 \hat{\varphi}^2(x) + a_4 \hat{\varphi}^4 + a_6 \hat{\varphi}^6 +}_{\text{local}}$$

$$+ \underbrace{c_2 |\nabla \hat{\varphi}|^2 + c_4 |\nabla \hat{\varphi}|^4 + \cdots}_{\text{gradients}}$$

$$- h(x) \hat{\varphi}(x) \Big\}$$

each local and gradients terms will have their scaling exponent

if we formally substitute

$$\hat{\varphi}(x) \to m(x)$$

i.e. the fluctuating $\hat{\varphi}$ by its self-consistent average we have the Landau theory -

Let's call

$$\begin{cases} x_2 \quad x_4 \quad \cdots \\ x^c_2 \quad x^c_4 \quad \cdots \end{cases}$$

the actual scaling exponent and

$$\begin{cases} x^0_2 \quad x^0_4 \quad \cdots \\ x^{c0}_2 \quad x^{c0}_4 \quad \cdots \end{cases}$$ their Landau counterpart

Since Landau gives a free energy (and non H) we have that a generalised Landau theory which takes into account more local and gradient terms will satisfy the previous relations

a) $d - 2x_\varphi^0 - x_2^0 = 0$ $\quad(x_a = x_2$, $x_\varphi = x_m)$

b) $d - 4x_\varphi^0 - x_4^0 = 0$

c) $d - 2 - 2x_\varphi^0 - x_2^{co} = 0$ gradient

d) $d - x_\varphi^0 - x_h^0 = 0$ local

further relations are

$$\left.\begin{aligned} &d - 6x_\varphi^0 - x_6^0 = 0 \\ &d - 8x_\varphi^0 - x_8^0 = 0 \end{aligned}\right\} \text{local}$$

$\vdots$

$$\left.\begin{aligned} &d - 2 - 4x_\varphi^0 - x_4^{co} = 0 \\ &d - 2 - 6x_\varphi^0 - x_6^{co} = 0 \\ &\vdots \end{aligned}\right\} \text{gradient}$$

if we assume $x_t = x_2^0 = 2$

we notice that

$$\begin{cases} \boxed{x_4^0 = 4-d} & >0 \text{ if } d<4 \\ x_6^0 = 6-2d & >0 \text{ if } d<3 \\ x_8^0 = 8-3d & >0 \text{ if } d<8/3 \\ \vdots & \end{cases}$$

$$\begin{cases} \boxed{x_2^{co} = 0} & \text{marginal} \\ x_4^{co} = 2-d & \\ x_6^{co} = 2(2-d) & >0 \text{ if } d<2 \\ x_8^{co} = 3(2-d) & \\ \vdots & \end{cases}$$

we thus have that in a generalised Landau theory for a scalar order parameter if $3<d<4$ only x_4^0 is relevant but x_2^{co} is marginal and should be taken into account. Now let's consider the behaviour of the actual scaling exponents...

Scaling holds at the actual critical point for $3 < d < 4$

assume that we start from $d=4$ and we want to move toward more "realistic" dimension $d<4$ then

$$x_k = (x_k^0)_{d=4} + \delta x_k$$

$$x_k^c = (x_k^{c0})_{d=4} + \delta x_k$$

for example

$x_4^0 = 4-d$ therefore

$$\boxed{x_4 = (x_4^0)_{d=0} + 4-d = \varepsilon}$$

but also

$$x_2 = (x_2^0)_{d=4} + \delta x_2$$

which means

$$\boxed{x_t = \left(\frac{d-2}{2}\right)_{d=4} + \delta x_2}$$

now let's consider x_φ itself

$$x_\varphi^0 = \frac{d-2}{2}$$

but actual scaling indexes should guarantee scaling properties (not fulfilled by Landau theory in $d<4$!!)

e.g. scaling law ⓐ

$$\eta = 2-d+2\beta/\nu$$

with $\beta = \frac{x_\varphi}{x_t}$ $\nu = \frac{1}{x_t}$

which means

$$\eta = 2-d+2x_\varphi$$

$$\boxed{x_\varphi = \underbrace{\frac{d-2}{2}}_{x_\varphi^0} + \frac{\eta}{2}}$$

To resume for a scalar order parameter and $3 < d < 4$
we can consider an Hamiltonian which has only 3 parameters

$$a_2 \quad a_4 \quad c_2$$

and we should perform an RG analysis on this 3 parameters space. As expected from previous considerations

$$c_2 \longrightarrow \bar{c}_2 = c_2$$

we are left with a 2 parameters flows

$$\begin{pmatrix} a_2 \\ a_4 \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{a}_2 \\ \bar{a}_4 \end{pmatrix}$$

and use $\varepsilon = 4 - \varepsilon$ as an expansion parameter