Wilson RG

operates in k-space
by pushing the infrared ($k \to 0$)
cutoff to zero

assumes no ultraviolet divergence
(i.e) theory is regular a $k \to \infty$
This can be obtained in a continuous
theory by a physical ultraviolet
cutoff

$$\Lambda \sim 1/a \leftarrow \text{lattice spacing}$$

1) integrate large momenta

$$\frac{\Lambda}{L} < k < \Lambda$$

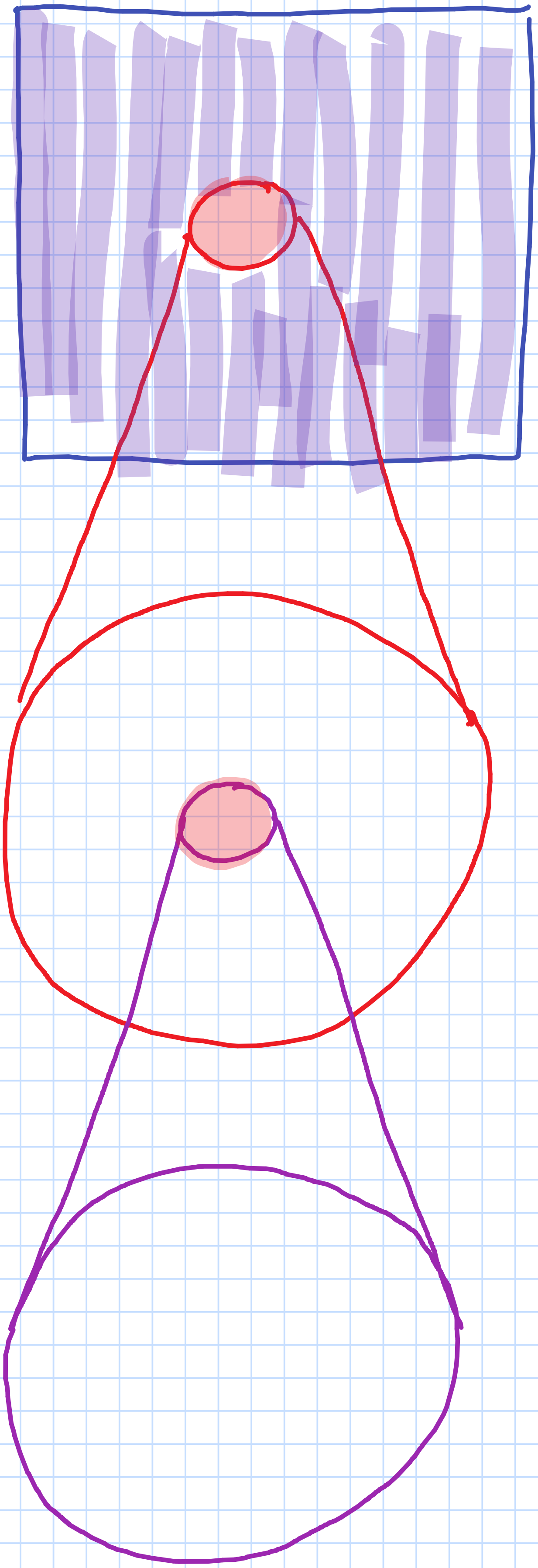
2) rescale small momenta

$$0 < k < \frac{\Lambda}{L} \longrightarrow 0 < q < \Lambda$$

$$q = Lk$$

3) rescale the fields

$$\hat{\varphi}_k \longrightarrow L^{d/2 - x_\varphi} \hat{\varphi}_{Lk}$$



step 1

step 2 + 3

Repeat

Field theory approach to 2nd order phase transitions

$$\mathcal{Z} = \int \mathbb{D}\hat{\varphi}\, e^{-H[\hat{\varphi}]}$$

we retain only up to 4th order in the scalar field $\hat{\varphi}(x)$ and second in the gradient

We use a standard notation

$$a_2 \to r_0/2 \quad a_4 \to u \quad e_2 \to c/2$$

$$H[\hat{\varphi}] = \int d^dx \left\{ \frac{1}{2} r_0 \hat{\varphi}^2_{(x)} + u\, \hat{\varphi}^4_{(x)} + \right.$$

$$\left. + c\, |\nabla \hat{\varphi}(x)|^2 - h(x)\, \hat{\varphi}(x) \right\}$$

To use Wilson-RG we must write H in k space to write H in k-space we allow for infrared cutoff by discretizing the $\vec{x}$ on a lattice therefore

$$\hat{\varphi}_k = \frac{1}{N} \Sigma_i \, e^{-i\vec{k}\cdot\vec{x}_i} \, \hat{\varphi}(\vec{x}_i)$$

$$\hat{\varphi}(\vec{x}_i) = \Sigma_{\vec{k}} \, e^{i\vec{k}\cdot\vec{x}_i} \, \hat{\varphi}_{\vec{k}}$$

such that for large volumes

$$\Sigma_{\vec{k}} \longrightarrow \frac{V}{(2\pi)^d} \int d^d k$$

$$\frac{1}{N} \Sigma_i \longrightarrow \int \frac{d^d x}{a^d}$$

$$\int d^d x \; \hat{\varphi}^2_{(x)} \rightarrow \Sigma_{\vec{k}} \; \hat{\varphi}_{\vec{k}} \, \hat{\varphi}_{-\vec{k}}$$

$$\int d^d x \, |\hat{\nabla} \varphi_{(x)}|^2 \rightarrow \Sigma_{\vec{k}} \, |\vec{k}|^2 \hat{\varphi}_k \, \hat{\varphi}_{-\vec{k}}$$

$$\int d^d x \; \hat{\varphi}^4_{(x)} \rightarrow \frac{1}{N} \sum_{k_1 k_2 k_3} \hat{\varphi}_{k_1} \hat{\varphi}_{k_2} \hat{\varphi}_{k_3} \hat{\varphi}_{-k_1-k_2-k_3}$$

$$\int d^d x \; h_{(x)} \varphi_{(x)} \rightarrow \Sigma_{\vec{k}} \; h_{\vec{k}} \, \varphi_{-\vec{k}}$$

$$\mathcal{H} = \frac{1}{2}\sum_{\vec{k}}\left(r_0 + c|\vec{k}|^2\right)\hat{\varphi}_{\vec{k}}\hat{\varphi}_{-\vec{k}}$$

$$+ \frac{u}{N}\sum_{k_1k_2k_3k_4} \delta_{k_1k_2k_3k_4}\,\hat{\varphi}_{\vec{k}_1}\hat{\varphi}_{\vec{k}_2}\hat{\varphi}_{\vec{k}_3}\hat{\varphi}_{\vec{k}_4}$$

$$- \sum_{k} h_{\vec{k}}\hat{\varphi}_{-\vec{k}}$$

momentum conservation

gaussian part

Wilson RG on Gaussian Field Teory

$$H_G = \frac{1}{2} \sum_{\vec{k}} \left(r_0 + c|\vec{k}|^2 \right) \hat{\varphi}_{\vec{k}} \hat{\varphi}_{-\vec{k}}$$

which is H evaluated at

$$u = 0$$
$$h = 0$$

let's define

complex vars

$$\operatorname{tr}_{\Lambda} (\ldots) = \int \prod_{\Lambda/L}^{\wedge} {}_{\vec{k}} \, d^2\varphi_{\vec{k}} \, (\ldots)$$

integration on large momenta

$$e^{-(H_L + F_L)} = \operatorname{tr}_{\Lambda} e^{-H_G}$$

e^{-F_L} is a constant induced by integration on large momenta

$$H_G = \frac{1}{2} \Sigma_k H_k \hat{\varphi}_k \hat{\varphi}_{-k}$$

$$H_k = (r_0 + c|\vec{k}|^2)$$

$$H_G = \underbrace{H_<}_{\sum_{0}^{\Lambda/L} k} + \underbrace{H_>}_{\sum_{\Lambda/L}^{\Lambda} k}$$

$$e^{-(H_L+F_L)} = e^{-H_<} \hat{\operatorname{tr}}\, e^{-H_>}$$

therefore

$$e^{-F_L} = \underbrace{\hat{\operatorname{tr}}\, e^{-H_>}}_{\text{gaussian integrals on}} = \prod_{\Lambda/L}^{\Lambda} {}_k \frac{2\pi}{H_{\vec{k}}}$$

gaussian integrals on complex vars φ_k

$$H_L = H_K = \frac{1}{2} \sum_{0}^{\Lambda/L}{}_k (r_0 + c|\vec{k}|^2) \hat{\varphi}_k \hat{\varphi}_{-k}$$

now we have to implement

step ② $\vec{q} = L\vec{k}$

step ③ $\hat{\varphi}_k \to \hat{\varphi}_{\underbrace{kL}_{q}} L^{d/2 - x_\varphi}$

$$H_L = \frac{1}{2} \sum_{0}^{\Lambda}{}_{\vec{q}} \left(r_0 + c\frac{|q|^2}{L^2}\right) L^{d-2x_\varphi} \hat{\varphi}_q \hat{\varphi}_{-q}$$

which have the same form of H_G provided

$$\bar{r}_0 = L^{d-2x_\varphi} r_0$$

$$\bar{c} = L^{d-2-2x_\varphi} c$$

we can express the scaling dimensions of $\bar{r}_0$ ($\bar{t}$) and $\bar{c}$ as a function of critical index η

$$2-\eta = d - 2x_\varphi$$

$$\boxed{\begin{cases} \bar{r}_0 = L^{2-\eta}\, r_0 \\ \bar{c} = L^{-\eta}\, c \end{cases}}$$

being $\eta \geq 0$ we have that
r_0 relevant if $\eta < 2$ and
marginal as $\eta = 2$
c irrelevant but marginal as $\eta = 0$

The gaussian FT has 2 kinds
of fixed points

$\boxed{\text{Trivial} \quad \eta = 2}$

$\boxed{\text{Gaussian} \quad \eta = 0}$

Trivial fixed points

$\eta = 2$ $\quad C(\vec{q}) \sim \frac{1}{q^{2-\eta}}$ does not diverge!

$\bar{r}_0 = r_0$

$\bar{c} = 0$

$$H^* = \frac{r_0}{2} \sum_{\vec{q}} \hat{\varphi}_q \hat{\varphi}_{-q}$$

if $r_0 > 0$ gaussian with finite fluctuations represent the trivial fixed point at $T = \infty$

if $r_0 < 0$ needs a finite u to represent the trivial fixed point at $T = 0$

Gaussian fixed point

$\eta = 0$ $\qquad C(\vec{q}) \sim \dfrac{1}{q^2}$ as in MFT

$\bar{r}_0 = r_0 = 0 \quad (T = T_c)$

$\bar{c} = c$

$$H^* = \frac{c}{2} \sum_q |q|^2 \hat{\varphi}_q \hat{\varphi}_{-q}$$

is non trivial fixed point which gives the Gaussian critical indexes (dimensionality dependent!)

$\beta = \dfrac{d-2}{4} \qquad x_t = x_{r_0} = 2 = \dfrac{1}{\nu}$

$\gamma = 1 \qquad \delta = \dfrac{d+2}{d-2}$

MFT critical indexes are obtained at the upper critical dimension $d=4$

Fluctuations at the fixed point

we can prove that whenever

$\eta < 2$ (i.e. $C(q)$ diverges as $q \to 0$)

Central Limit Theorem (CLT) is not valid for the local random variable $\hat{\varphi}(x)$ -

Consider a discrete lattice and the spatial average

$$\hat{\varphi}^{(N)} = \frac{1}{N} \sum_i \hat{\varphi}(x_i)$$

does it coincides with the ensemble average

$$\langle \hat{\varphi}(\bar{x}_1) \rangle$$

on a generic point x_1 ?

to see this let's calculate the variance

$$\langle \hat{\varphi}(N)^2 \rangle = \frac{1}{N^2} \sum_{ij} \langle \hat{\varphi}(x_i) \hat{\varphi}(x_j) \rangle$$

$$= \frac{1}{N^2} \sum_i \sum_{\vec{r}} \langle \hat{\varphi}(\vec{x}_i) \hat{\varphi}(\vec{x}_i + \vec{r}) \rangle$$

now assume that on a short range ℓ the correlation function

$$C(\vec{r}) = \langle \hat{\varphi}(\bar{x}) \hat{\varphi}(\bar{x} + \vec{r}) \rangle$$

saturates to its factorized value i.e

if $|\vec{r}| \gg \ell$ $\quad C(\vec{r}) \simeq \langle \varphi(\bar{x}) \rangle^2$

indep on $\bar{x}$

since ℓ is finite in the thermodynamic limit then

$$\langle \hat{\varphi}^{(N)2} \rangle \simeq \frac{1}{N^2} \underbrace{N}_{\sum_i} \underbrace{N}_{\sum_r} \langle \hat{\varphi} \rangle^2$$

Now at the critical point $\langle \hat{\varphi} \rangle = 0$ but

$$\langle \hat{\varphi}(x)\, \hat{\varphi}(\vec{x}+\vec{r}) \rangle = C(|\vec{r}|) \quad (\text{large } r)$$

$$C(r) \sim L^{-2x_\varphi}\, C\left(\frac{r}{L}\right)$$

$$\langle \hat{\varphi}^{(N)2} \rangle = \frac{1}{N^2} \sum_i \sum_{\vec{r}} L^{-2x_\varphi}\, C\left(\frac{r}{L}\right)$$

as $L \to \infty$ Const $\sim C(0)$

$$\langle \hat{\varphi}^{(N)2} \rangle \propto \frac{1}{N^2} N N L^{-2x_\varphi} = \frac{1}{N} L^{d-2x_\varphi}$$

$$\langle \hat{\varphi}^{(N)2} \rangle \propto \frac{1}{N} L^{2-\eta}$$

while CLT states

$$\langle \hat{\varphi}^{(N)2} \rangle \propto \frac{1}{N}$$

and $N = L^d$

Stability of the Gaussian fixed point

to check the stability of the Gaussian fixed point we must linearise the transformation around the Gauss FP

$$R_L(H_G^* + \tau\,\delta H) = R_L(H_G^*) + \tau\, L_L(H_G^*)\,\delta H$$

or in terms of representation of L_L

$$\underbrace{\bar{p}_i - p_i}_{\delta p_i} = \sum_j \underbrace{\left(\frac{\partial f_i^{(L)}}{\partial p_j}\right)}_{(\mathbb{L}_L)_{ij}} \delta p_j$$

find eigenvalues/vector of

$$(\mathbb{L}_L)_{ij}$$

In our case the Gauss FP has $(h=0)$

$$\begin{pmatrix} r_0^* \\ u^* \\ c^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

In the parameter's space of the full Hamiltonian we can thus proceed with a

Perturbation Expansion

to determine

$$\begin{pmatrix} \bar{r}_0 - 0 \\ \bar{u} - 0 \\ \bar{c} - c \end{pmatrix} = \Big[\quad \mathbb{L} \quad \Big] \begin{pmatrix} r_0 - 0 \\ u - 0 \\ c - c \end{pmatrix}$$

we see that only the 2 parameters space (r_0, u) is relevant

Perturbation Expansion in RG (classical fields...)

$$e^{-(H_L+F_L)} = \underset{\wedge}{\mathrm{tr}}\, e^{-(H_0+\tau\delta H)}$$

now when $\tau=0$

$$e^{-(H_L^{(0)}+F_L^{(0)})} = \underset{\wedge}{\mathrm{tr}}\, e^{-H_0}$$

therefore if

$$H_L+F_L = H_L^{(0)}+F_L^{(0)}+\delta(H_L+F_L)$$

$$e^{-\delta(H_L+F_L)} = \langle e^{-\tau\delta H}\rangle_\wedge$$

where $\langle(\ldots)\rangle_\wedge = \dfrac{\underset{\wedge}{\mathrm{tr}}\, e^{-H_0}(\ldots)}{\underset{\wedge}{\mathrm{tr}}\, e^{-H_0}}$

This gives rise to the ordinary cumulant expansion

$$\delta(H_L + F_L) = \tau \langle \delta H \rangle_\lambda +$$

$$+ \frac{\tau^2}{2} \left\{ \langle \delta H^2 \rangle_\lambda - \langle \delta H \rangle_\lambda^2 \right\}$$

$$+ \mathcal{O}(\tau^3)$$

Perturbation expansion around the Gaussian fixed point

let

$$H_0 = \frac{c}{2} \sum_k |k|^2 \hat{\varphi}_k \hat{\varphi}_{-k} \quad \text{(massless)}$$

this is a gaussian Hamiltonian for which we will apply the Wick's theorem to calculate to first order the average of

$$\tau \, \delta H = \underbrace{\frac{r_0}{2} \sum_k \hat{\varphi}_k \hat{\varphi}_{-k}}_{\delta H^{(2)}} + \underbrace{\frac{U}{N} \sum_{k_1, k_2, k_3} \hat{\varphi}_{k_1} \hat{\varphi}_{k_2} \hat{\varphi}_{k_3} \hat{\varphi}_{-k_1-k_2-k_3}}_{\delta H^{(4)}}$$

- $\delta H^{(2)}$ quadratic
- $\delta H^{(4)}$ quartic

to calculate the average of quadratic & quartic terms we need the propagator

$$\langle \hat{\varphi}_k \hat{\varphi}_{k'} \rangle_\Lambda = \frac{\delta_{k+k'}}{c|k|^2} \quad \frac{\Lambda}{L} < (k, k') < \Lambda$$

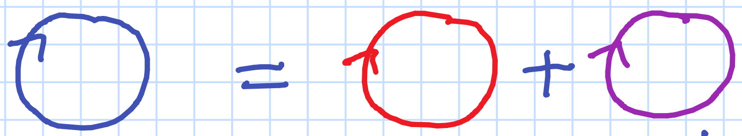
which is the Fourier transform of correlation function in the Gaussian FT

1) Quadratic part

$$\langle \delta H^{(2)} \rangle_\Lambda = \underbrace{\frac{\bar{r}_0^{(2)}}{2} \sum_{0\,q}^{\Lambda} \hat{\varphi}_q \hat{\varphi}_{-q}}_{\delta H_L^{(2)}} + \underbrace{\frac{r_0}{2} \sum_{\Lambda/L\,k}^{\Lambda} \langle \hat{\varphi}_k \hat{\varphi}_{-k} \rangle}_{\delta F_L^{(2)}}$$

$\boxed{\bar{r}_0^{(2)} = L^{2-\eta} r_0}$ after rescaling

this partial average can be viewed diagrammatically by associating to Σ_k appearing on ● a loop



i.r. modes gives rise to $\delta H_L^{(2)}$

integrated u.v. modes $\delta F_L^{(2)}$

the sum over $\vec{q}$ & $\vec{k}$ is expressed by a closed loop. Red modes are not averaged while violet one are.

2) Quartic Part

Now write the quartic part as

$$\frac{U}{N}\sum_{\substack{k_1\\k_2\\k_3\\k_4}} \underbrace{\delta_{k_1+k_2+k_3+k_4,0}}_{\text{interaction vertex}} \hat{\varphi}_{k_1}\hat{\varphi}_{k_2}\hat{\varphi}_{k_3}\hat{\varphi}_{k_4}$$

The interaction vertex express the k conservation. We can fulfill the conservation by rewriting this term as

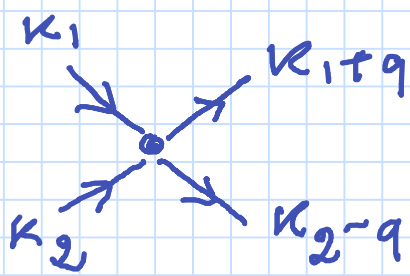
$$\boxed{\frac{U}{N}\sum_{k_1,k_2,q}\hat{\varphi}^*_{k_1+q}\,\hat{\varphi}^*_{k_2-q}\,\hat{\varphi}_{k_2}\,\hat{\varphi}_{k_1}}$$

Since $\varphi^*_k = \varphi_{-k}$ then

$$-k_1-q-k_2+q+k_1+k_2=0$$

as expressed by the δ function

Graphically



Now we have to average the interaction term which is the product of 4 complex variables

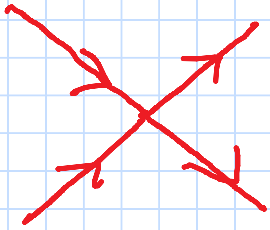
let us point out that for a single complex variable $z = x + iy$

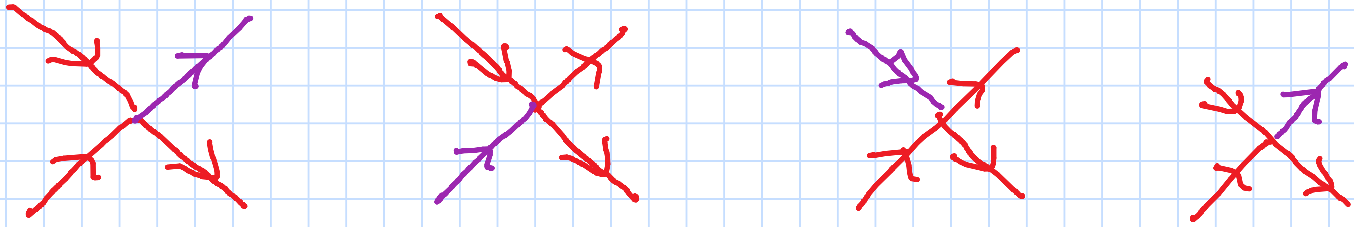
$$\int dx\,dy\; e^{-\frac{|z|^2}{2\sigma^2}} (z^*)^2 = \int dx\,dy\; e^{-\frac{|z|^2}{2\sigma^2}} (z)^2 = 0$$

since
$$(z^*)^2 = x^2 - y^2 - 2ixy$$
$$(z)^2 = x^2 - y^2 + 2ixy$$

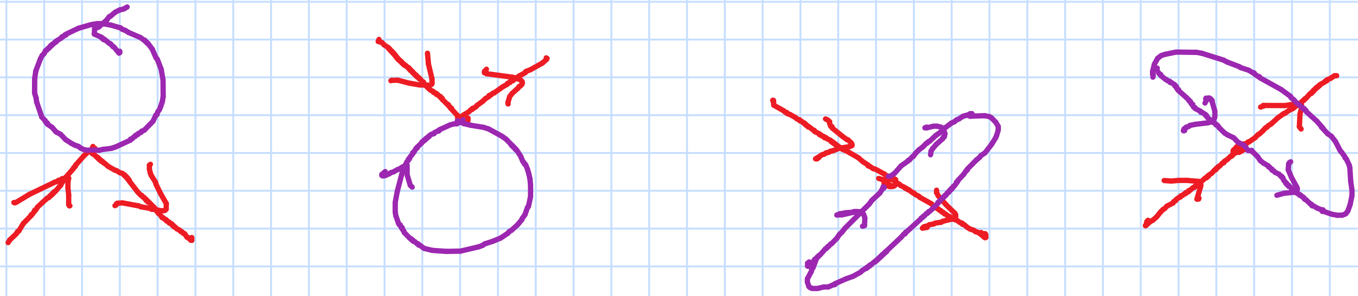
therefore as we express the quartic product as ★ Wick's theorem states that we have to contract pairs of φ and φ^* (and not $\varphi^*\varphi^*$ or $\varphi\varphi$) in all possible way

In the diagram ★ the average is carried out by connecting only inbound and out bound legs only

$\langle \delta H^{(4)} \rangle =$  contributes to H_L

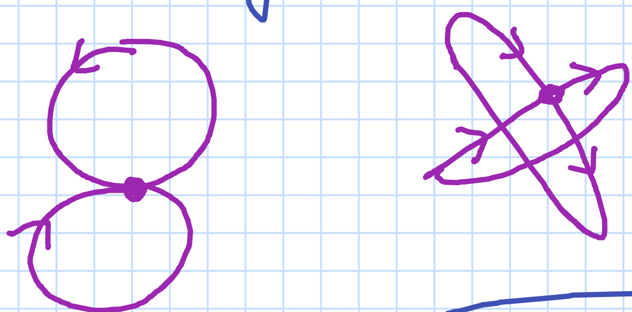


these are zero since a single φ has zero average

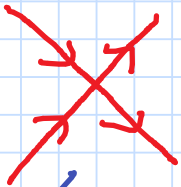
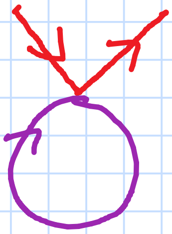


renormalizes quadratic term of i.r. modes

third order terms in —— are absent because they have vanishing average



Contributes to F_L

$$\langle \delta H^{(4)} \rangle =$$  $$+ 4$$  $$+ \delta F_L^{(4)}$$

constant

renormalizes quartic term

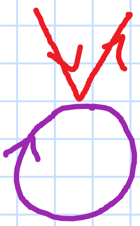
renormalizes quadratic term

$$\frac{u}{N}\sum_{0,\{k\}}^{\Lambda/L} \varphi^*\varphi^*\varphi\varphi = \frac{u}{N L^d}\sum_{0,\{q\}}^{\Lambda} L^{2d} L^{-4x_\varphi}\varphi^*\varphi^*\varphi\varphi$$

where for $a=1$ $N=V$ and $V=\bar{V} L^d$

from this term we have

$$\boxed{\bar{u} = L^{d-4x_\varphi}\, u = L^{4-2\eta-d}\, u}$$

 $$= \frac{u}{N}\underbrace{\sum_{0,q}^{\Lambda} L^{d-2x_\varphi}\varphi_q^*\varphi_q}\;\underbrace{\sum_{\Lambda/L,\,k}^{\Lambda}\frac{1}{c|k|^2}}$$

$$\frac{1}{N}\sum_{\Lambda/L,\,k}^{\Lambda}\frac{1}{c|k|^2} = \frac{\cancel{V}}{\underbrace{\cancel{N}}_{\text{as } a=1}(2\pi)^d}\,\Omega_d\int_{\Lambda/L}^{\Lambda} dk\, k^{d-1}\,\frac{1}{ck^2}$$

Ω_d d-dim solid angle

$$\frac{4}{N}\sum_{\Lambda/L}^{\Lambda}{}_{k}\frac{1}{c|k|^2}=\frac{\Omega_d}{(2\pi)^d}\frac{4\Lambda^{d-2}}{d-2}(1-L^{d-2})=B_d(1-L^{d-2})$$

$$\langle \delta H^{(4)}\rangle=\frac{\bar{u}}{\bar{N}}\sum_{q_1q_2q}\varphi^*_{q_1+q}\varphi^*_{q_2-q}\varphi_{q_2}\varphi_{q_1}+$$

$$\boxed{+\,u\,L^{d-2\chi_\varphi}B_d\left(1-L^{d-2}\right)\sum_q\varphi^*_q\varphi_q}$$

$$+\,\delta F_L^{(4)}$$

renormalizes r_0

Collecting togheter results

from $\langle\delta H^{(2)}\rangle$

from quartic part of $\langle\delta H^{(4)}\rangle$

from quadratic part of $\langle\delta H^{(4)}\rangle$

$$\bar{r}_0=\underbrace{L^{2-\eta}r_0}-\underbrace{L^{2-\eta}B_d(1-L^{2-d})u}$$

$$\bar{u}=\underbrace{L^{4-2\eta-d}u}$$

$$\begin{pmatrix} \overline{r}_0 \\ \overline{u} \end{pmatrix} = \begin{bmatrix} L^2 & B_d(L^2-L^{4-d}) \\ 0 & L^{4-d} \end{bmatrix} \begin{pmatrix} r_0 \\ u \end{pmatrix}$$

where we have evaluated the matrix at the Gaux FP $\eta=0$ taking into account that $r_0^*=u^*=0$ the matrix is a representation of

$$\mathbb{L}_L = \begin{bmatrix} L^2 & B_d(L^2-L^{4-d}) \\ 0 & L^{4-d} \end{bmatrix}$$

eigenvalues

$$\lambda_{r_0} = L^2 \qquad \lambda_u = L^{4-d}$$

eigenvector

$$\underline{V}_{r_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \underline{V}_u = \begin{pmatrix} -B_d \\ 1 \end{pmatrix}$$

with $B_d > 0$ as $d > 2$

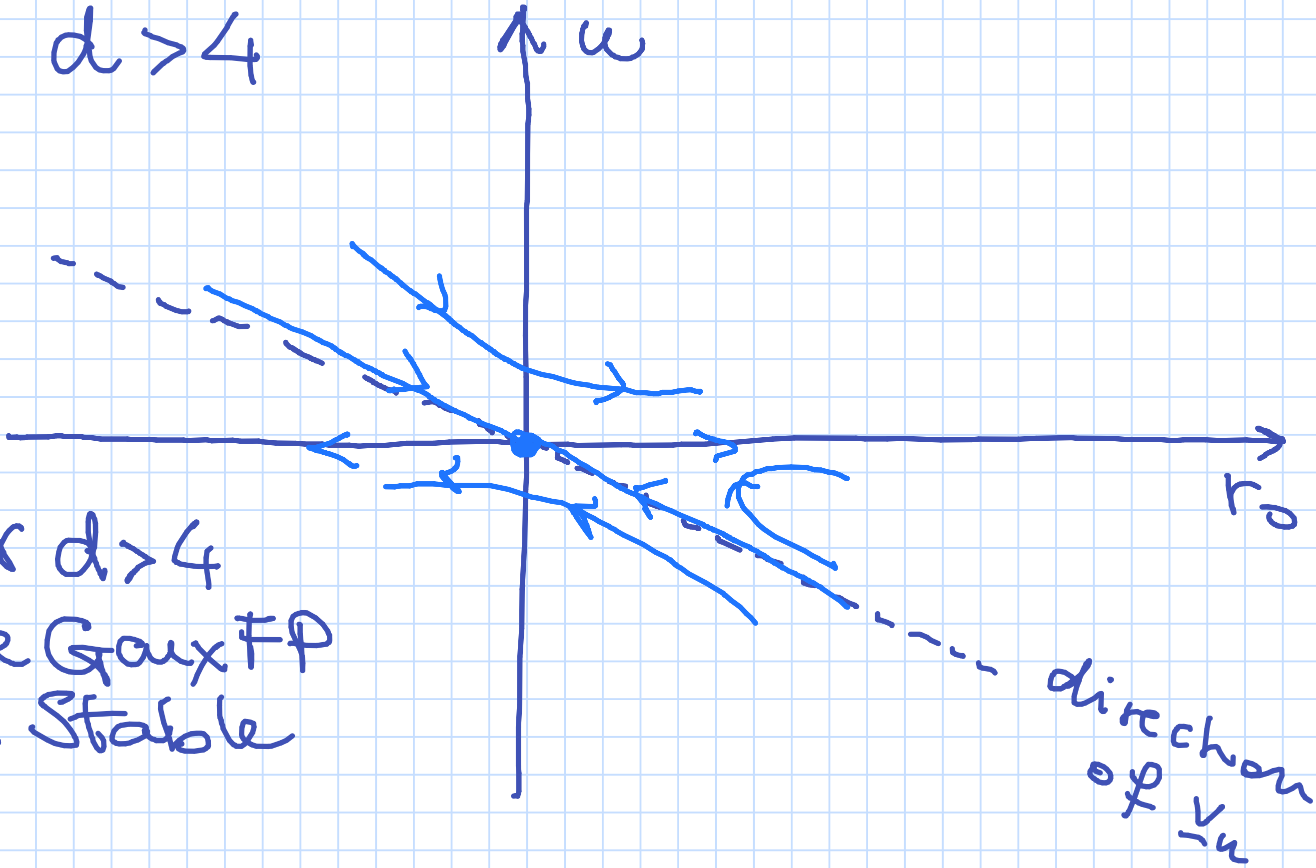
then the linearized flow of RG around the Gaux FP can be drawn since

direction of $\underline{V}_{r_0}$ is associated with $\lambda_{r_0} = L^{2} > 0$ is always unstable

direction of $\underline{V}_u$ is stable for $d > 4$ $(\lambda_u = L^{4-d} < 0)$
unstable for $d < 4$

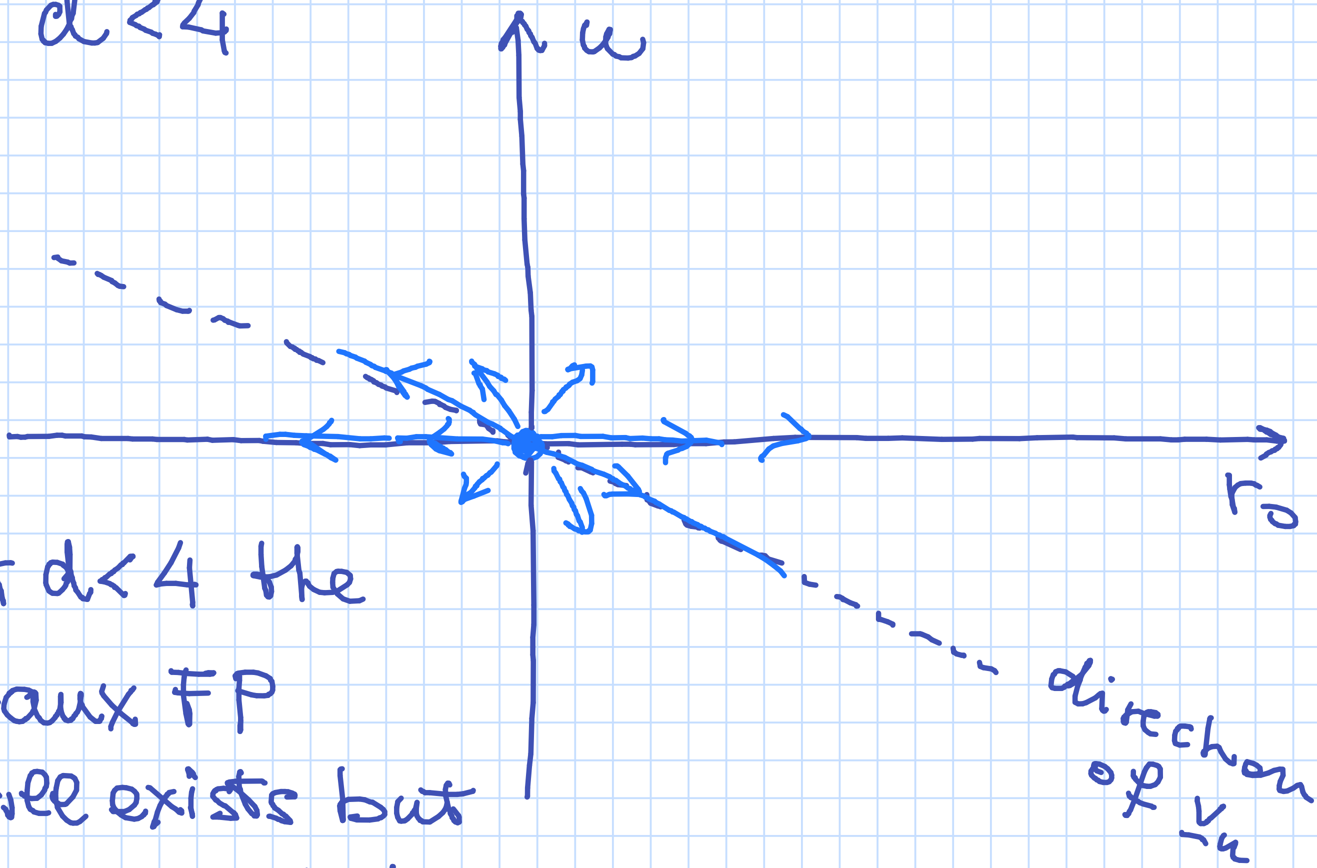
a more stable fixed point should appear in realistic dimensions $d < 4$

$d > 4$



for $d>4$ the Gaux FP is stable

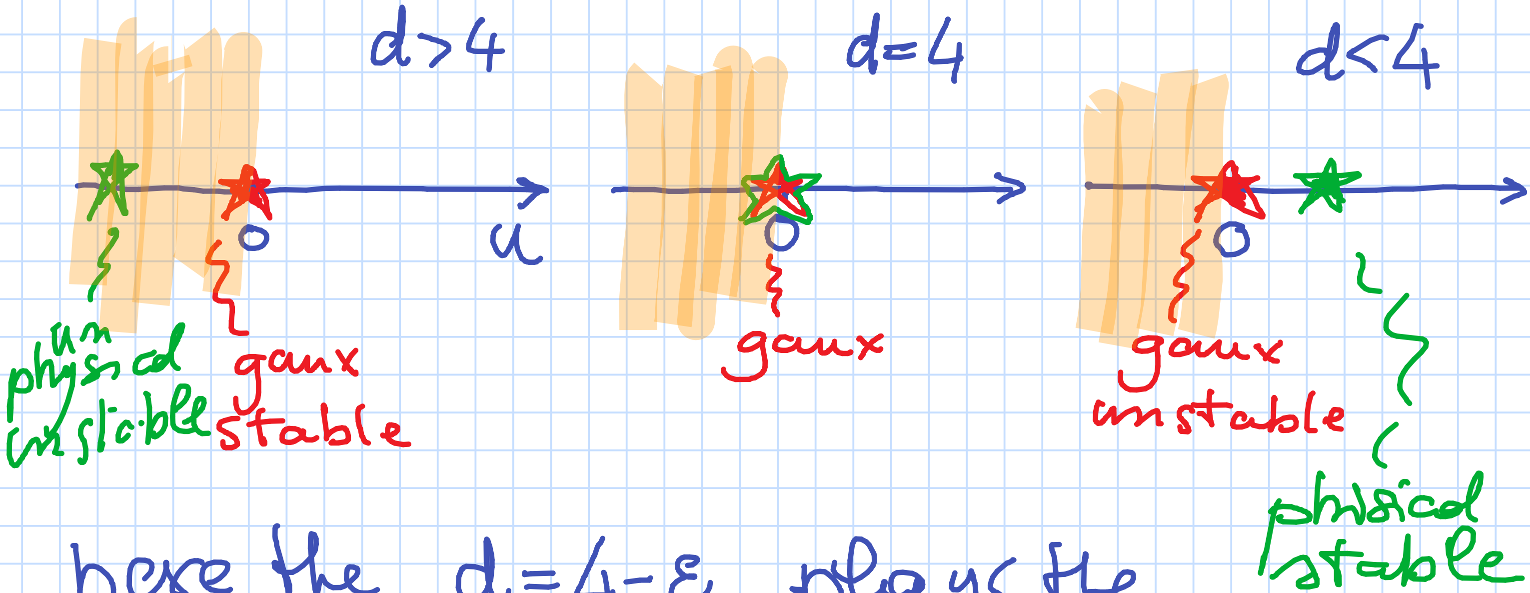
$d < 4$



for $d<4$ the Gaux FP still exists but it is unstable

Wilson-RG and the 4-ε expansion

The main idea of Wilson & Kogut is that as far a $d \lesssim 4$ another fixed point ★ which becomes stable exists near the Gaux FP



here the $d = 4 - \varepsilon$ plays the role of a continuous parameter being ε small

Perturbation theory should be performed up to 2nd order which allows to a finite corrections to the critical parameters (e.g. T_c)

for Gauss FP

$r_0^* = 0 \qquad u^* = 0$

including second order corrections a new fixed point arises with

$r_0^* \sim \varepsilon$ $\qquad \varepsilon = 4-d$

$u^* \sim \varepsilon$

$d < 4$