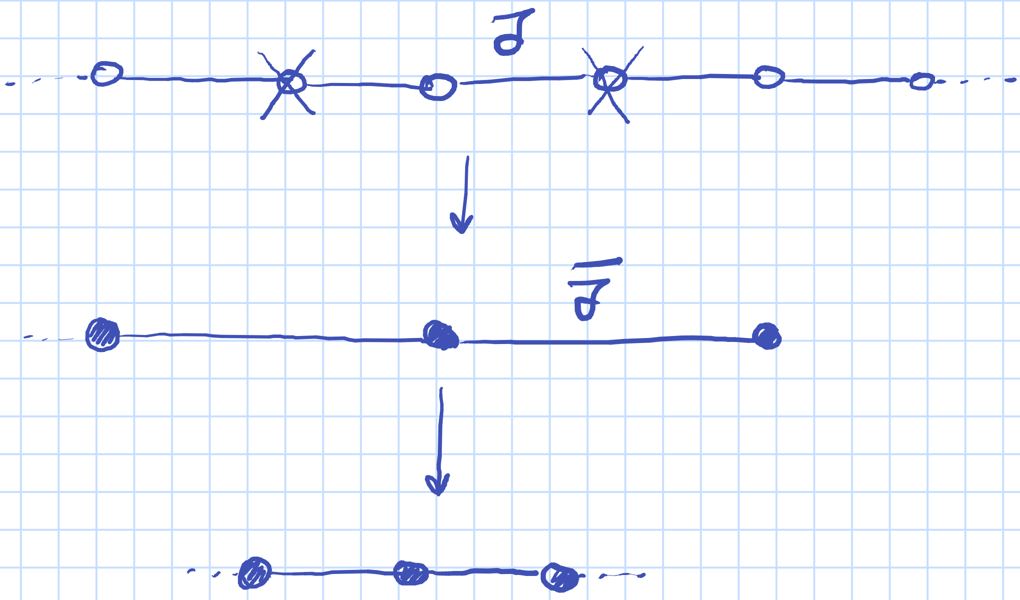
Renormalization Group approach to 1d Ising model (from Chandler chap. 5.6)

Kadanoff real space approach via "decimation" procedure



Using transfer matrix

$$Z = \mathrm{tr}\, \mathbb{T}^N$$

$$\mathbb{T} = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} = e^{\beta J}\mathbb{1} + e^{-\beta J}\sigma_x$$

for $h = 0$ (critical condition)

Now one can integrate out the ~~x~~ spins defining

$$\bar{T}(\sigma_1\sigma_2) = \mathrm{tr}_{\sigma'}\, T(\sigma_1\sigma')\,T(\sigma'\sigma_2)$$

therefore

$$\bar{T} = \mathbb{T}^2 = e^{2\beta J}\,\mathbb{1} + e^{-2\beta J}\underbrace{\sigma_x^2}_{\mathbb{1}} + 2\sigma_x$$

$$\bar{T} = 2\,\mathrm{ch}(2\beta J)\,\mathbb{1} + 2\,\sigma_x$$

the scaling hypotesis assumes for $\bar{T}$ the same form of T i.e.

$$\boxed{\bar{T} = A(L)\,T}$$

Constant depending on the scale (here $L=2$)

here the parameter is J which will be scaled to a new param $\bar{J}$

by comparing T & $\overline{T}$ we get

$$A e^{\beta \overline{J}} = 2 \operatorname{ch}(2\beta J) \qquad (1)$$

$$A e^{-\beta \overline{J}} = 2 \qquad (2)$$

in such a way that

$$\overline{T} = A e^{\beta \overline{J}} \mathbb{1} + A e^{-\beta \overline{J}} \sigma_x = A\, T(\beta \overline{J})$$

we can obtain an equation for $\overline{J}$ by substituting (2) in (1)

$$e^{2\beta \overline{J}} = \operatorname{ch}(2\beta J)$$

$$\beta \overline{J} = \frac{1}{2} \ln[\operatorname{ch}(2\beta J)]$$

The Renormalisation Group flow equations are defined as

$$J = g(\overline{J})$$

as in the scaling theory we want to determine the physical observable e.g. the order parameter

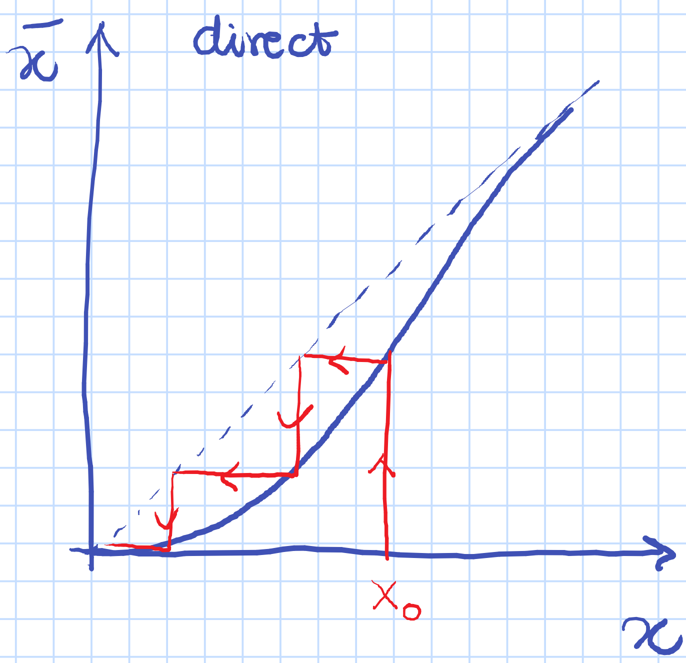
$\mathcal{M}(t,h)$ by scaling the parameters with the scale L (reverse)

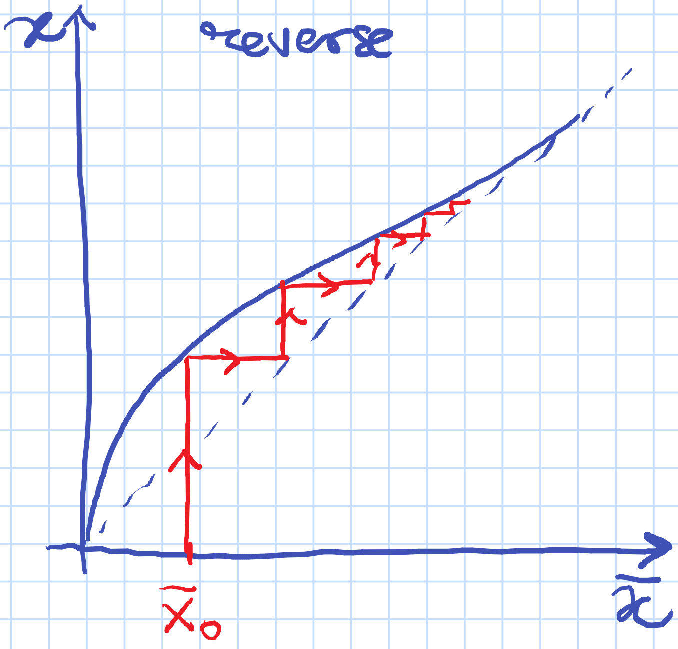
we here go from the knowledge of the parameter at large scale to the knowledge of the parameter at shortest scale (reverse)

if

$$\bar{x} = \beta \bar{J} \qquad x = \beta J$$

$$\bar{x} = \frac{1}{2} \ln[ch(2x)] \qquad x = \frac{1}{2} ch^{-1}(e^{2\bar{x}})$$





An iteration scheme can be constructed using either direct or reverse scheme i.e.

direct

$$\bar{x}_{n+1} = \frac{1}{2} \ln ch(2\bar{x}_n)$$

reverse

$$x_{n+1} = \frac{1}{2} ch^{-1}(e^{2x_n})$$

the flow of these 2 iterations is drawn in red in the previous page

there are thus 2 FIXED POINTS of these recursions

direct $\bar{x} \to 0$

reverse $x \to \infty$

let's consider direct scaling:

by removing half of the spin at each step we mapped the system to one with smaller interaction this fact indicates that for any finite temperature there is no long range order induced by a finite interaction J

In other words the system at large scale is seen as a non interacting system
and looks paramagnetic

the two fixed points

$$\beta J = 0 \qquad \beta J = \infty$$

are called trivial fixed points

Now consider the reverse iteration scheme. In this scheme we assume a finite large scale coupling x_0

RG flow gives in this case a perfectly ordered $(x \to \infty)$ small scale behaviour. This is the $T=0$ $(\beta \to \infty)$ limit where even in $d=1$ we may have a non zero order parameter

Now let us consider the free energy per spin

$$f = -\frac{1}{N\beta} \ln Z$$

$$Z = \operatorname{tr} T^N = A^{N/2} \operatorname{tr} T(\bar{J})^{N/2}$$

$$f(\beta J, N) = -\frac{1}{2\beta} \ln A + \frac{1}{2} \bar{f}(\beta \bar{J}, \bar{N})$$

with $\bar{N} = N/2$

this resembles the scaling for scalar order parameters

$$f(t\ h) = \frac{1}{L^d} \bar{f}(\bar{t}, \bar{h})$$

now $L = 2$ $d = 1$ here we should include $\bar{J}$ in the definition of $\bar{f}$...

$$f(\beta J, N) = \frac{1}{2}\left(\bar{f}(\beta \bar{J}, \bar{N}) - \bar{J}\right) - \frac{1}{2\beta} \ln 2$$