Stability

(a)

$$e^{-(H_L+F_L)} = \operatorname*{tr}_{\Lambda} e^{-H} = \operatorname*{tr}_{\Lambda} e^{-(H_0+\lambda H_1)}$$

up to second order in $H_1$

cumulant expansion $\qquad \operatorname*{tr}_{\Lambda} = \sum_{\frac{\Lambda}{L}<k<\Lambda}$

closed sol systems

$$\operatorname*{tr}_{\Lambda} e^{-H_0}\left(\sum_m \frac{H_1^m}{m!}\right)$$

$$H_L+F_L = -\ln \operatorname*{tr}_{\Lambda} e^{-(H_0+H_1)}$$

$$H_L^{(0)}+F_L^{(0)}+\lambda(H_L^{(1)}+F_L^{(1)})+\lambda^2(H_L^{(2)}+F_L^{(2)})$$

$$= -\ln \operatorname*{tr}_{\Lambda} e^{-H_0}\left(1+\lambda H_1+\frac{\lambda^2}{2}H_1^2\right)$$

$$\ln(1+x_1+x_2^2) = 1+(x_1+x_2^2)-\frac{(x_1+x_2^2)^2}{2}$$

$$= 1+x_1+x_2^2-\frac{x_1^2}{2} = 1+x+\frac{x^2}{2}$$

$$e^{-(H_0+F_L^{(0)})}\; e^{\lambda(H^{(1)}+F_L^{(1)}+\cdots}\; 1+x_{1/2}$$

$$e^{-\lambda(H^{(1)}+F_L^{(1)})-\lambda^2(H^{(2)}+F_L^{(2)})} = \frac{\operatorname*{tr}_{\Lambda} e^{-H_0+\lambda H_1}}{e^{-H_0^{(1)}}\, e^{-F_L^{(0)}}}$$

$$e^{-\lambda(H^{(1)}+F_L^{(1)})-\lambda^2(H_L^{(2)}+F_L^{(2)})} = \frac{\operatorname*{tr}_{\Lambda} e^{-H_0^{\wedge}}\, e^{\lambda H_1}}{\operatorname*{tr}_{\Lambda} e^{-H_0^{\wedge}}}$$

$$e^{-(H_L+F_L)} = \underset{\wedge}{\mathrm{tr}}\, e^{-(H_0+\lambda H_1)}$$

fast modes

$$H_L + F_L = H_L^{(0)} + F_L^{(0)} + \delta H$$

$$e^{-(H_L^{(0)}+F_L^{(0)})} = \underset{\wedge}{\mathrm{tr}}\, e^{-H_0}$$

$$-\delta H = \ln \frac{\underset{\wedge}{\mathrm{tr}}\, e^{-H_0+\lambda H_1}}{\underset{\wedge}{\mathrm{tr}}\, e^{-H_0}} = \ln \langle e^{-\lambda H_1} \rangle_\wedge$$

$$\ln \langle e^{-\lambda H_1} \rangle_\wedge = \ln \{ 1 \mp \lambda \langle H_1 \rangle + \frac{\lambda^2}{2} \langle H_1^2 \rangle \}$$

$$\simeq -\lambda \langle H_1 \rangle + \frac{\lambda^2}{2} \langle H_1 \rangle^2 - \frac{\lambda^2}{2} \langle H_1^2 \rangle$$

$$\delta H = \lambda \langle H_1 \rangle + \frac{\lambda^2}{2} \left( \langle H_1^2 \rangle - \langle H_1 \rangle^2 \right)$$ cumulant expans.

$$H^* = c \sum_{0\,q}^{\wedge} |q|^2 \varphi_q \varphi_{-q}$$

(c)

$$H_1 = \frac{r_0}{2}\Sigma_k \varphi_k \varphi_{-k} + \frac{u}{L^d}\Sigma\, \varphi_k \varphi_{k'} \varphi_{k''}\, \varphi_{-k-k'-k''}$$

$H_1^{(2)}$ $\quad$ $H_1^{(4)}$

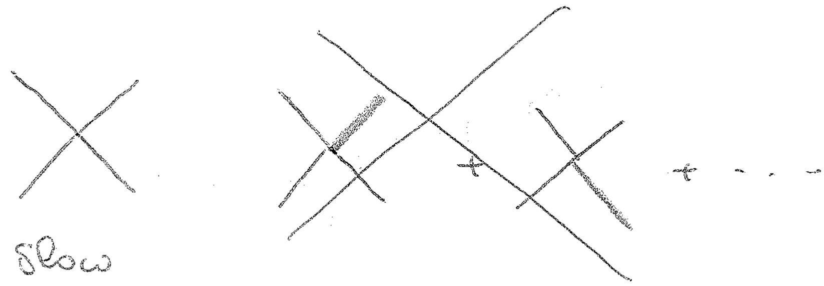
$$\langle H_1^{(2)} \rangle_\Lambda = \frac{1}{2}\bar{r}_0 \sum_{0}^{q} \hat{\varphi}_q \hat{\varphi}_{-q} + \sum_{\Lambda/L}^{\Lambda} {}_k \ln \frac{2\sigma}{r_0}$$

constant
due to integration
on gaussian

$$\langle \hat{\varphi}_k \hat{\varphi}_{k'} \rangle = \frac{\delta(k+k')}{c|k|^2}$$

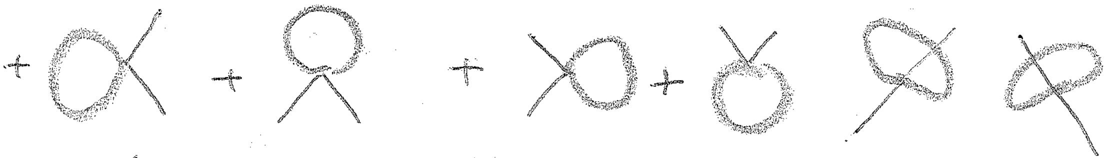
$$\langle H_1^{(2)} \rangle = \frac{1}{2}\bar{r}_0 \sum_{0}^{q} \hat{\varphi}_q \hat{\varphi}_{-q} + \frac{1}{2} r_0 \sum_{\Lambda/L}^{\Lambda} {}_k \langle \varphi_k \varphi_{-k} \rangle$$

$$H_1^{(4)} = \frac{u}{\cancel{V}} \sum_{k_1 k_2 k_3 k_4} \delta_{k_4, -k_1-k_2-k_3}\, \hat{\varphi}_{k_1} \hat{\varphi}_{k_2} \hat{\varphi}_{k_3} \hat{\varphi}_{k_4}$$

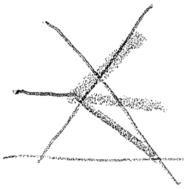


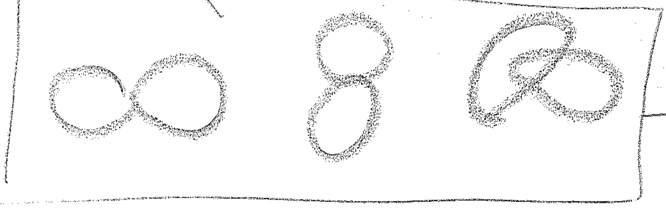
slow $\quad$ fast

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4\cdot 3\cdot 2}{4} = 6$$



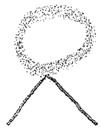
$6 = \binom{4}{2}$





$\to$ constant

d

$\langle H_1^{(4)} \rangle_\Lambda =$  $+\ 6$  $+ \underbrace{F^{(4)}}_{\text{constant}}$

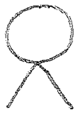
low modes

$$\frac{u}{V}\sum_{0<k_1k_2k_3k_4<\frac{\Lambda}{L}} \varphi\varphi\varphi\varphi = \frac{u}{VL^d}\sum_{0<q_1q_2q_3q_4<\Lambda} \overbrace{L^{2d}\, L^{-4x_\varphi}}^{2(2-\eta)}\varphi\varphi\varphi\varphi$$

$$= \frac{\bar{u}}{\bar{V}}\sum_{0<\{q\}<\Lambda}\varphi\varphi\varphi\varphi$$

$$\bar{V} = \frac{V}{L^d} \qquad \bar{u} = u\, L^{d-4x_\varphi} = L^{4-2\eta-d}\, u$$

$$d-2x_\varphi = 2-\eta$$

 $= \dfrac{u}{\bar{V} L^d}\displaystyle\sum_{0<q<\Lambda} L^{d-2x_\varphi}\varphi_q\varphi_{-q}\sum_{\frac{\Lambda}{L}<k<\Lambda}\frac{1}{c|k|^2}$

$d-2x_\varphi = 2-\eta$

$2(2-\eta)-d$

$$\Omega_d\int_{\Lambda/L}^{\Lambda} dk\; k^{d-2-1}$$

$$\Omega_d\,\frac{\Lambda^{d-2}-(\Lambda/L)^{d-2}}{d-2} =$$

$$\frac{\Omega_d\Lambda^{d-2}}{d-2}\left(1-L^{2-d}\right)$$

total rescaling

$$\text{---}\!\!\bigcirc\!\!\text{---} = \frac{u L^{2-\eta}}{V L^{d}} \left[ \sum_{q} \varphi_q \varphi_{-q} \right] \frac{\Omega_d \Lambda^{d-2}}{d-2} \left(1 - L^{2-d}\right)$$

$$\frac{1}{V} \sum_{\frac{\Lambda}{L} < k < \Lambda} \frac{1}{c |k|^2} = \frac{\Omega_d \Lambda^{d-2}}{d-2} \left(1 - L^{2-d}\right)$$

so that

$$\begin{cases} \bar{r}_0 = L^{2-\eta} r_0 + L^{2-\eta} u B_d \left(1 - L^{2-d}\right) \\ \bar{u} = L^{4-2\eta-d} u \end{cases}$$

$$\bar{c} = c L^{-\eta}$$

The gaussian fixed point set $\eta = 0$

$$\bar{r}_0 = L^2 r_0 + u B_d \left(L^2 - L^{4-d}\right)$$

$$\bar{u} = L^{4-d} u$$

$$\bar{c} = c$$

$$\begin{pmatrix} \bar{r}_0 \\ \bar{u} \\ \bar{c} \end{pmatrix} = \begin{pmatrix} L^2 & B_d \left(L^2 - L^{4-d}\right) & 0 \\ 0 & L^{4-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

la rappresentazione di RG attorno ad $H^* = H_G$ è quindi

$$L_L = \begin{pmatrix} L^2 & (L^2 - L^{4-d}) B_d & 0 \\ 0 & L^{4-d} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

con $B_d > 0$ se $d > 2$

autovalori

$$(\lambda - L^2)(\lambda - L^{4-d}) = 0$$

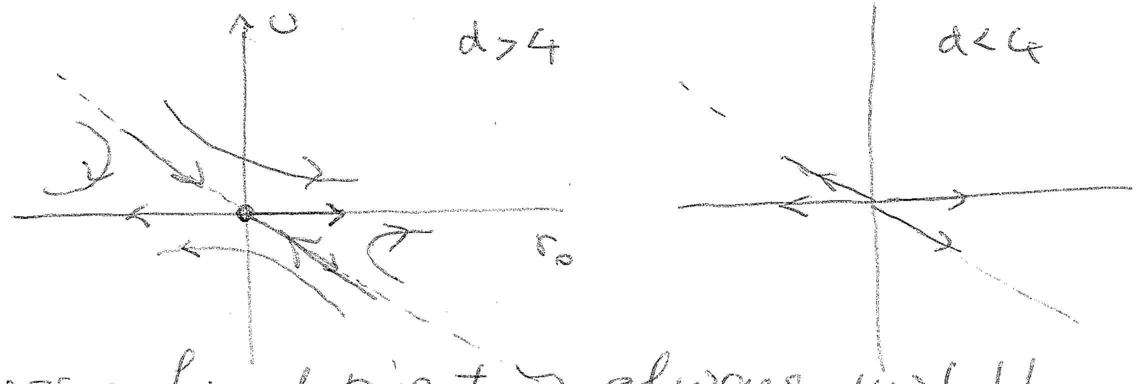
$\lambda_{r_0} = L^2 \quad \lambda_u = L^{4-d}$

$r_0$ sempre rilevante direzione instabile

$u$ rilevante solo se $d < 4$ direzione

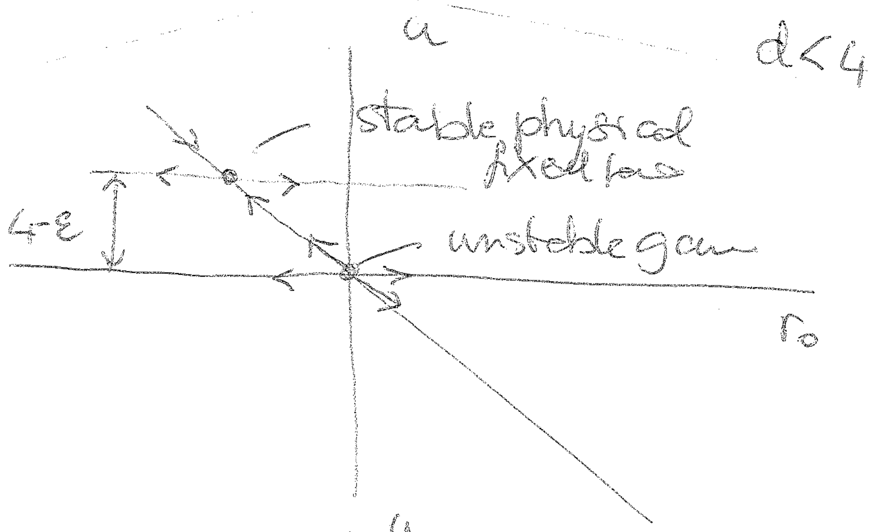
autovettori

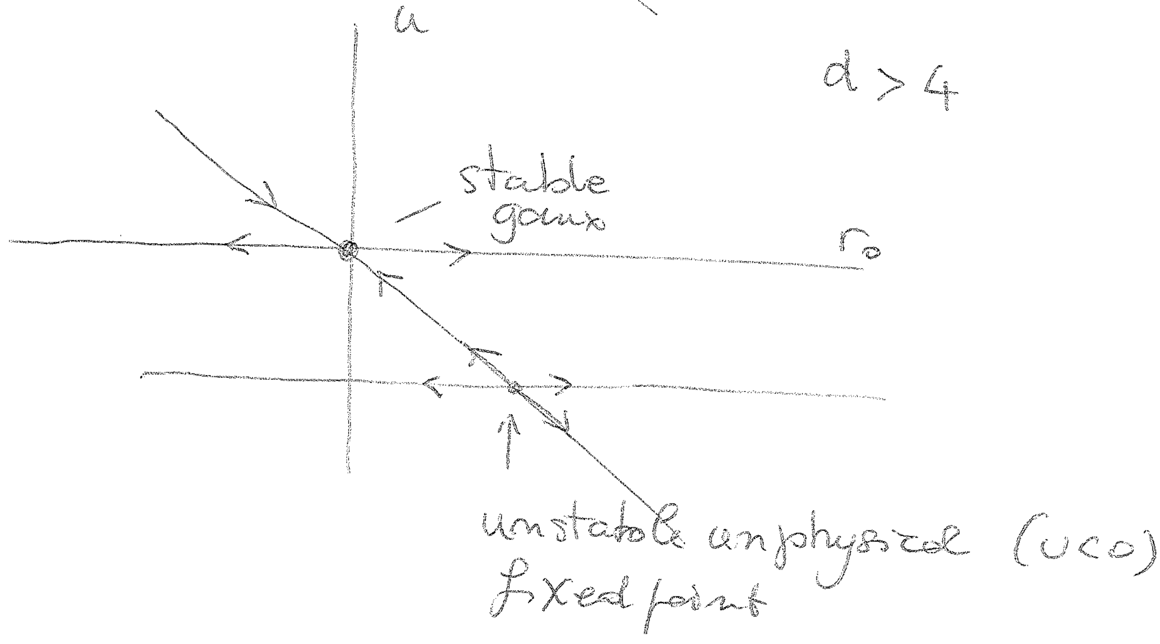
$$u_{r_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_u = \begin{pmatrix} -B_d \\ 1 \end{pmatrix} \frac{1}{\sqrt{1 + B_d^2}}$$




gaussian fixed point is always instable

as $d < 4$


u
d<4
stable physical
fixed pas
4-ε
unstable gau
$r_0$


u
d>4
stable
gaux
$r_0$
unstable unphysical (u<0)
fixed point