Exercise #3

due date: 19th November 2019

a) Consider the stochastic equation for the moment of a particle under the action of external random forces $\xi(t)$ (in one dimension):

$$\dot{p}(t) = -\gamma p(t) + \xi(t)$$

where

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\xi(t') \rangle = 2M\gamma k_b T\delta(t-t')$$

- Derive without approximation the average mean square displacement

$$\Delta\left(t\right) = \langle |x\left(t\right) - x\left(0\right)|^2 \rangle$$

- Derive the behaviour of $\Delta\left(t\right)$ for large and small times and define the time scale above which the behaviour of $\Delta\left(t\right)$ is **linear** in time.

b) A polymer can be constructed as a three dimensional random walk where the position of the n+1-th monomer is given by

$$\vec{r}_{n+1} = \vec{r}_n + a\hat{u}_{n+1}$$

where a is the monomer spacing and $\hat{u}$ is a random unit vector. The length of the polymer made by $N+1$ monomers can be estimated by

$$\ell = \sqrt{\langle |\vec{r}_N - \vec{r}_0|^2 \rangle}$$

where the average is taken over the random orientation of the unit vectors $\hat{u}_n$.

- Calculate the ratio ℓ / Na and comment the result.

c) Prove that the entropy S is given by $S = -k_B tr\rho \log\rho$ where ρ is the equilibrium density matrix in the canonical ensemble. First perform the calculation in the classical canonical ensemble, then generalize it to the quantum case.

d) Consider the Fokker-Planck equation in one dimension

$$\frac{\partial}{\partial t}P\left(x,t\right) = \frac{\partial}{\partial x}F\left(x\right)P\left(x,t\right) + \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}P\left(x,t\right)$$

following the lines of the notes

show that for a potential problem

$$F\left(x\right) = -\frac{\partial}{\partial x}V\left(x\right)$$

the stationary distribution is given by

$$P\left(x\right) = \exp\left(-2V\left(x\right)/\varepsilon\right).$$

Derive the Maxwell Boltzmann distribution as stationary distribution of the momenta and of the position. For the position distribution use the overdamped approximation in the corresponding Langevin equation.
