1 Exercises: set 1

1. Consider the the Minkowski metric written in Cartesian coordinates

$$ds^2 = dt^2 - \delta_{ij}\, dx^i dx^j \,.$$

   Write the same metric in spherical coordinates.

2. Let p be the 4-momentum of a particle of mass m. Express m in terms of p.

3. Write the components of the 4-momentum p of a particle of mass m in a reference frame where the particle has standard velocity $\vec{\mathbf{v}}$.

4. Given two timelike 4-vectors A and B, the sum $A + B$ will still be a timelike vector ?

5. Consider a generic covariant vector A_μ, show that $\mathbf{A}^2 = A_\mu A_\nu \eta^{\mu\nu}$ is Lorentz invariant and compute its expression explicitly in terms of the components.

6. Let $\mathbf{p}$ be the 4-momentum of particle, show that

$$E_u = \mathbf{p} \cdot \mathbf{V}$$

   is the energy measured by an observer with 4-velocity $\mathbf{V}$. Hint: consider first the case where $\mathbf{V}$ is the 4-velocity of the particle $\mathbf{U}$ and then $\mathbf{V}$ is the 4-four velocity of an observer in the lab frame where the particle has standard velocity $\vec{\beta}$. Notice that no Lorentz transformation is needed.

7. Write the components of the 4-momentum p of a massless particle in a reference frame where it moves in the direction $\vec{\mathbf{n}}$:

8. In the Lab frame a photon (massless particle) is emitted by a moving source with ordinary velocity $\vec{\mathbf{v}}$ in a direction $\vec{\mathbf{n}}$. Without the use of Lorentz tranformations find the ratio between the energy of the photon measured in the Lab frame E_{rec} and the energy E_{em} of the photon measured in the emitter frame (relativistic Doppler effect).

9. Consider the action of a Lorentz transformation in the x direction on a 4-vector, starting from that find the the explicit form for a boost in a arbitrary direction $\vec{\mathbf{n}}$ and finally determine its infinitesimal form.

10. Consider a generic non-null 4-vector u, such that $u^2 = \pm 1$ in Minkowski space, verify that

$$P^\mu_\nu = \delta^\mu_\nu - u^\mu\, u_\nu$$

is a projector tensor that projects in the subspace orthogonal to u. Consider the case where $u^2 = 1$ and represents the 4-velocity of a particle and choose a comoving reference frame; write the components of the projectors in such a frame.

11. Consider the equation of motion for a relativistic particle of mass written in a Lorentz covariant form
$$m\frac{d\mathbf{U}}{d\tau} = \mathbf{K}\,,$$
where $\mathbf{U}$ is the particle 4-velocity and τ its proper time and $\mathbf{K}$ is the 4-force. What kind of 4-vector is the acceleration, timelike, spacelike or null-like ? Show that U is orthogonal (by using the Minkowski metric) to the 4-acceleration and use that to find the the time component of the 4-force in the lab frame and to determine the time derivative of the energy and momentum of the particle.

12. Consider the differential operator box defined as
$$\Box = \eta^{\mu\nu}\,\partial_\mu\partial_\nu\,,$$
in "Cartesian coordinates" (t, x^1, x^2, x^3). Compute its action on a scalar (Lorentz invariant) function f. Show that $\Box$ is a Lorentz invariant differential operator.