1 Exercises: set 2

1. Show that, given the 4-dimensional domain $V = \{t \in [t_1, t_2], \quad \mathbf{x}^2 \leq R^2\}$ and a generic vector field A^μ, the following identity holds

$$\int_V d^4x \, \partial_\mu A^\mu = \int_{\partial V} d^3x \, n_\mu A^\mu \, .$$

Determine explicitly the unity normal n_μ of each part of the boundary $\partial V$of V.

2. Consider the Lagrangian $\mathcal{L}(X, \phi)$ for a scalar field ϕ, with $X = \partial_\mu \phi \, \partial^\mu \phi$. Find the Euler-Lagrangian equation for ϕ in terms of the derivative of $\mathcal{L}$ with respect to X denotated by $\mathcal{L}_X$ and the derivative of $\mathcal{L}$ denoted by $\mathcal{L}_\phi$.

3. Consider a free scalar field of mass m interaction with a classical source J described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \phi^2 + J \, \phi = \mathcal{L}_0 + J \, \phi \, .$$

In particular consider J of the form

$$J = \left[j_1 \, \delta^{(3)}(\vec{x} - \vec{x}_1) + j_2 \, \delta^{(3)}(\vec{x} - \vec{x}_2) \right] \; ;$$

with j_1, j_2 constants. Show that

$$< 0| T e^{i \int d^4x \, \phi \, J} |0 >= e^{-i \, T \, V(r)} \, ;$$

where and taking T is a very large time, finding the form of the potential $V(r)$. The result can be interpreted by saying that static sources for a field experience a force due to the exchange of virtual particles described by $V(r)$.

4. Take two free left spinors with canonical kinetic terms. Write the most general Lagrangian $\mathcal{L}$ and discuss its symmetries. Find a field redefinition such that the mass matrix is diagonal together with the kintetic matrix. For what form of the mass matrix $\mathcal{L}$ is symmetric under a $U(1) \sim SO(2)$?

Now consider the following mass matrix

$$\begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \, ;$$

where $M >> m$. In such limit find the physical masses. Set $m = 100$ GeV and suppose that given Lagrangian $\mathcal{L}$ captures the key features of the free Lagrangian for the electronic neutrino ν_e. For what value of M the ν_e mass scale is reproduced ? Is the form of the mass matrix compatible with $U(1)$ invariance of $\mathcal{L}$? The solution is an illustration of the so called "see-saw" mechanism.