An Introduction to Geometrical Methods in Physics

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Contents

1 Introduction

The present set of short notes are intended to give an introduction to the basic notion of differential geometry and tensor analysis. To keep the matter as simple as possible we shall consider an open set U of $\mathbb{R}^n$ without give a formal definition of a manifold. Most of the results can be extended at the local level to a n-dimensional manifold. The main goal is to give a geometric definition of vectors, 1-forms and tensors developing the basics of exterior calculus.
References

- A nice a modern geometric approach to tensor calculus can be found in the first couple of chapters in
Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics by Thorne and Blandford; Princeton University Press. A draft version is available for free on the web.
See also the first chapters of the classic textbook
Gravitation by Misner, Thorne and Wheeler; Wiley.

- A introduction to differential geometry can be found in
Geometrical Methods of Mathematical Physics by B. Shutz; Cambridge University Press.

2 Manifolds

3 Vectors and 1-Forms

The standard definition of a vector given as an "arrow-like" object connection two points applis only in the case of an affine Euclidean space. Consider a smooth [1] function in a open set $U \subset \mathbb{R}^n$, namely $f : U \to \mathbb{R}$ and a Jordan curve $\gamma : [a,\, b] \to U$ passing in the point $x_0 \in U$ with $\gamma(t_0) = x$. A vector can be consider as the tangent vector $\boldsymbol{\xi}$ tangent to γ in x_0. We can take the directional derivative of f along $\boldsymbol{\xi}$, namely

$$\boldsymbol{\xi}(f)_{|x} = \frac{d}{dt} f \cdot \gamma_{|x} \,. \tag{1}$$

From the definition is clear that such an action is linear in f. Thus, we can define a vector as linear machine that takes in a function in U and produce e number. Given a set of coordinates $\{x^i,\ i = 1, 2, \cdots n\}$ for the points of U; we have for γ

$$\gamma : [a, b] \to x^i(t)\,; \qquad i = 1, 2, \cdots n\,; \tag{2}$$

then

$$\boldsymbol{\xi}(f)_{|x_0} = \sum_{i=1}^{n} \frac{dx^i(t)}{dt}_{|t_0} \frac{\partial f}{\partial x^i} \equiv \frac{dx^i(t)}{dt}_{|t_0} \partial_i f\,; \tag{3}$$

in the second line we have introduced the Einstein convention on repeated indicies that will be used throughout the present notes. By considering a different curve γ' passing in x_0 will get a different vector $\boldsymbol{\xi'}$ in x the space of vectors in x_0 is a real linear space of dimension n. Notice that given two functions f_1 and f_2 on U, we have that

$$\boldsymbol{\xi}(a\, f_1 + b\, f_2)_{|x_0} = a\,\boldsymbol{\xi}(f_1)_{|x_0} + b\,\boldsymbol{\xi}(f_2)_{|x_0}\,; \tag{4}$$

where $a, b \in \mathbb{R}$; in the same way for two vectors $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$

$$\left(a\boldsymbol{\xi} + b\boldsymbol{\xi}'\right)(f)_{|x_0} = a\,\boldsymbol{\xi}(f)_{|x_0} + b\,\boldsymbol{\xi}'(f)_{|x_0}\,. \tag{5}$$

A special class of curves are obtained by considering the i-th coordinates curve:

$$\gamma_i : [a, b] \to \{x^1 = x_0^1,\, \cdots, x^{i-1} = x_0^{i-1},\, x^i = t,\, x^{i+1} = x_0^{i+1},\, \cdots, x^n = x_0^n\}\,. \tag{6}$$

where $t \in [a, b]$ and the coordinates of x_0 are $\{x_0^i\}$. We have that that tangent vector $\boldsymbol{\xi}_i$ to such a curve in x_0 acts on f as

$$\boldsymbol{\xi}_i(f)_{|x_0} = \partial_i f_{|x_0}\,. \tag{7}$$

[1] A smooth function is an infinitely differentiable function; for us it just means differentiable as many times as needed.

The linear space of vectors in x is called tangent space of U and denoted by $T(U)_x$ and, when a set of coordinates is introduced, is spanned by the n vectors

$$\boldsymbol{\xi}_i \equiv \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}\,, \qquad i = 1, \cdots, n\,. \tag{8}$$

As a result a generic vector $\boldsymbol{\xi} \in T(U)_x$ tangent to a curve γ in x_0 can be written as

$$\boldsymbol{\xi} = \xi^i \, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}\,; \tag{9}$$

where ξ^i are the coordinates of the tangent vector to γ in x_0

$$\xi^i = \frac{dx^i(t)}{dt}_{|t_0}\,. \tag{10}$$

A vector is a geometric object and its existence is independent from a coordinate system. One can easily finds how the components ξ changes under a change of coordinates $x^i \to x'^i$

$$\boldsymbol{\xi}(f) = \xi^i \frac{\partial f}{\partial x^i} = \xi^i \, \frac{\partial x'^j}{\partial x^i} \, \frac{\partial f}{\partial x'^i} \equiv \xi'^j \, \frac{\partial f}{\partial x'^i}\,; \tag{11}$$

thus

$$\boldsymbol{\xi} = \xi^i \, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}} = \xi^i \, \frac{\partial x'^j}{\partial x^i} \, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x'^i}} \Rightarrow \xi'^i = \xi^i \, \frac{\partial x'^j}{\partial x^i}\,. \tag{12}$$

A vector field is a smooth assignment of vectors in the points of U. It convient to consider the union of the various tangent spaces in the points of U, namely $T(U) = U_x \times T(U)_x \sim \mathbb{R}^n \times U$; such a space is called cotangent bundle of U. It is worth to stress is not mandatory to use a coordinate basis for writing a vector. Indeed, consider a *generic* basis of vectors $\{\boldsymbol{E}_i\}$ which can be expressed in terms of the coordinate basis as

$$\boldsymbol{E}_i = \Lambda_i^j \, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}\,; \tag{13}$$

where the $n \times n$ Λ invertible matrix describes the change of basis and will depend on x in general. It should be stressed that by no means there exists a coordinates system $\{y^i\}$ such that

$$\boldsymbol{E}_i = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y^i}}\,; \tag{14}$$

thus the above relation is in general false, see the problems section.

3.1 Problems

1. Given the following vector in $\mathbb{R}^3$ defined in cartesian coordinates (x, y, z)

$$\boldsymbol{\xi} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} + 2\frac{\boldsymbol{\partial}}{\boldsymbol{\partial z}}$$

   find its expression in spherical coordinates.

2. Given two vectors fields $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, show that for any function f

$$[\boldsymbol{\xi}, \boldsymbol{\zeta}]f = -[\boldsymbol{\zeta}, \boldsymbol{\xi}]f = \boldsymbol{\xi}(\boldsymbol{\zeta}(f)) - \boldsymbol{\zeta}(\boldsymbol{\xi}(f))$$

   is a linear operator on f which defines a new vector field $\boldsymbol{Z} = [\boldsymbol{\xi}, \boldsymbol{\zeta}] = \mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{\zeta}$. called the Lie derivative of $\boldsymbol{\zeta}$ with respect to $\boldsymbol{\xi}$. Find the components Z_μ of $\boldsymbol{Z}$ in a generic coordinates system.

3. Show that for a coordinates basis of vectors we have that

$$\left[\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}}\right] = 0\,;$$

   while for a generic basis of vectors $\{\boldsymbol{E}_i\}$

$$[\boldsymbol{E}_i,\, \boldsymbol{E}_j] \neq 0\,.$$

4 1-Forms

Given vector space V one can always construct ist dual space made of all linear functional of V, namely the maps $\mathcal{L} : V \to \mathbb{R}$. The linear space of 1-forms in x denoted by $T(U)^*_x$ is the dual space of the tangent space in x and is called the cotangent space in x. Thus, a 1-form $\boldsymbol{\omega}$ is a linear functional that takes a vector, say $\boldsymbol{\xi}$ and send it into a real number:

$$\boldsymbol{\omega}(\boldsymbol{\xi}) \in \mathbb{R}\,; \tag{15}$$

popular alternative notations are

$$< \boldsymbol{\omega}, \boldsymbol{\xi} >= i_{\boldsymbol{\xi}}\boldsymbol{\omega}\,. \tag{16}$$

Given a set of coordinates $\{x^i\}$ one can define the a set basis 1-forms $\{\boldsymbol{dx}^i\}$ dual of $\{\frac{\partial}{\partial x^i}\}$ by

$$\boldsymbol{dx}^i(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}}) = \delta^i_j\,, \qquad i,j = 1,\cdots,n\,. \tag{17}$$

A generic 1-form can be written in terms of basis $\{\boldsymbol{dx}^i\}$ as

$$\boldsymbol{\omega} = \omega_i \boldsymbol{dx}^i\,; \tag{18}$$

ω_i are the components. We have that We have that

$$\boldsymbol{\omega}(\boldsymbol{\xi}) = \omega_i\,\boldsymbol{dx}^i\left(\xi^j\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}}\right) = \omega_i\,\xi^j\,\boldsymbol{dx}^i\left(\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}}\right) = \xi^i\,\omega_i\,; \tag{19}$$

for a vector $\boldsymbol{\xi}$ of components ξ^i; such a quantity is independent from the choice of coordinates as we will show shortly.

The different notation for the components of vectors and 1-forms underlines the fact that they have different transformation properties under a change of coordinates. Indeed, by changing coordinates $x^i \to x'^i$ we have that

$$\boldsymbol{\omega} = \omega_i\,\boldsymbol{dx}^i = \omega_i\,\frac{\partial x^i}{\partial x'^j}\,\boldsymbol{dx'}^j \equiv \omega'^j\,\boldsymbol{dx'}^j \qquad \Rightarrow \omega'^i = \omega_j\,\frac{\partial x^i}{\partial x'^j}\,. \tag{20}$$

Thus, while the transformation properties of vector components involve the Jacobian matrix of the change of coordinates, the 1-form components involve the inverse of such a matrix. In particular As a result we have that

$$\boldsymbol{\omega}(\boldsymbol{\xi}) = \xi^i\,\omega_i = \xi'^i\,\omega'_i \tag{21}$$

as promised. The contraction of a 1-form with a vector is independent from the choice of coordinates.

As for vectors one can also consider a basis of 1-forms that is not related to the standard basis of 1-forms provided by a system of coordinates.

A for vectors, a smooth assignment of 1-form in U gives rise to a field of 1-forms in U. It convient to consider the union of the various cotangent spaces in the points of U, namely $T(U)^* = U_x \times T(U)^*_x \sim \mathbb{R}^n \times U$; such a space is called cotangent bundle of U.

5 Tensors

Tensors can be obtained by taking the multilinear tensor product of 1-forms and vectors. Take for instance a couple of vectors $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ and consider

$$\boldsymbol{T} = \boldsymbol{\xi} \otimes \boldsymbol{\zeta}\,, \tag{22}$$

in a coordinate basis

$$\boldsymbol{T} = \xi^i\,\zeta^j\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}} \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}}\,. \tag{23}$$

Such a geometric object is a linear machine that accepts as input a couple of 1-form $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ to produce a number, namely

$$\boldsymbol{T}(\boldsymbol{\omega}, \boldsymbol{\Omega}) = \xi^i \omega_i\,\zeta^j \Omega_j\,. \tag{24}$$

More in general a tensor a generic tensor with two slots for two 1-forms has the following form

$$\boldsymbol{T} = T^{ij}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}} \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}}\,. \tag{25}$$

The most In general tensor is a multilinear machine with p ordered slots for 1-forms and q ordered slots for vectors, namely

$$\boldsymbol{T} = T^{i_1\cdots i_p}_{j_1\cdots j_q}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^{i_1}}} \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^{i_p}}} \otimes \boldsymbol{dx^{j_1}} \otimes \cdots \boldsymbol{dx^{j_q}}\,; \tag{26}$$

such a tensor is called of type (p, q) and accepts p 1-forms (p ordered vector-like upper indices) and q ordered 1-form (q ordered 1-form-like lower indices). Upper vector-like indices are called controvariant and the lower ones covariant. Thus a tensor of of type (p, q) is multilinear map of the form

$$\boldsymbol{T} : TU^* \otimes \cdots \otimes TU^* \otimes TU \otimes \cdots \otimes TU \to \mathbb{R}\,. \tag{27}$$

The trasformation properties of the components of $\boldsymbol{T}$ are easily derived from the corresponding trasformation of vectors (12) and 1-forms (21); we have that

$$T'^{i_1\cdots i_p j}_{\;\;j1\cdots j_q} = T^{a_1\cdots a_p}_{b_1\cdots b_q}\,\frac{\partial x'^{i_1}}{\partial x^{a_1}}\,\frac{\partial x'^{i_2}}{\partial x^{a_2}}\cdots\frac{\partial x'^{i_p}}{\partial x^{a_p}}\,\frac{\partial x^{b_1}}{\partial x'^{j_1}}\,\frac{\partial x^{b_2}}{\partial x'^{j_2}}\cdots\frac{\partial x^{b_q}}{\partial x'^{j_q}}\,. \tag{28}$$

Of course one consider just a partial contraction of a tensor; for instance consider the case of a $(1, 1)$ tensor

$$\boldsymbol{T} = T^i_j\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}} \otimes \boldsymbol{dx^j}\,. \tag{29}$$

Then, the contraction with a vector $\boldsymbol{\xi}$ gives a $(1,0)$ tensor, a vector,

$$\boldsymbol{T}(\cdot,\boldsymbol{\xi}) = T^i_j\,\xi^j\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}\,. \tag{30}$$

While the contraction with a 1-form $\boldsymbol{\omega}$ gives a $(0,1)$ tensor, a 1-form,

$$\boldsymbol{T}(\boldsymbol{\omega},\cdot) = T^i_j\,\omega_i\,\boldsymbol{dx^j}\,. \tag{31}$$

6 Metric Tensor

Consider the case when a tensor $\boldsymbol{g}$ of type $(0,2)$ exists with the following properties

- $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{v},\boldsymbol{u})$; symmetry
- $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) = 0\ \forall \boldsymbol{v} \in T(U)$ implies that $\boldsymbol{u} = 0$; non degeneracy.

In a coordinate basis we have that

$$\begin{aligned} &\boldsymbol{g} = g_{ij}\,\boldsymbol{dx^i} \otimes \boldsymbol{dx^j} \\ &\text{with } g_{ij} = g_{ji}\,, \qquad \det(g_{ij}) \neq 0\,. \end{aligned} \tag{32}$$

The above conditions implies that as a matrix g_{ij} has no vanishing eigenvalue. When all eigenvalues are positive one speaks of a Riemannian metric or pseudo-Riemannian if this is not the case. two important physical examples are the following. The metric can be used to defin a scalar $\cdot$ product for vectors according to

$$\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})\,, \qquad \boldsymbol{u},\,\boldsymbol{v} \in T(U)\,. \tag{33}$$

In a coordinate basis where

$$\boldsymbol{u} = u^i\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}\,, \qquad \boldsymbol{v} = v^i\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}\,; \tag{34}$$

we have that

$$\boldsymbol{u}\cdot\boldsymbol{v} = g_{ij}u^i\,v^j\,. \tag{35}$$

Besides a definition of a scalar product a metric maps naturally a vector $\boldsymbol{(}u)$ into the following 1-from $\tilde{\boldsymbol{u}}$

$$\tilde{\boldsymbol{u}} = \boldsymbol{g}(\cdot,\boldsymbol{u}) = g_{ij}u^j\,\boldsymbol{dx^i}\,. \tag{36}$$

Also the inverse is true. By using the non-degeneracy of the metric one can define the inverse metric tensor $\tilde{g}$ defined by

$$\tilde{\boldsymbol{g}} = g^{ij}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}} \otimes \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^j}}\,, \qquad g^{ik}g_{kj} = \delta^i_j\,. \tag{37}$$

Thus given a 1-form $\boldsymbol{\omega} = \omega_i\,\boldsymbol{dx^i}$ we can construct a vector $\tilde{\boldsymbol{\omega}}$ by

$$\tilde{\boldsymbol{\omega}} = \tilde{\boldsymbol{g}}(\cdot,\boldsymbol{\omega}) = g^{ij}\omega_j\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^i}}\,. \tag{38}$$

Thus, given a metric vectors and 1-forms can be used interchangeably.

Two important examples of a metric structure are the following. Consider $\mathbb{R}^3$ covered by a global set of cartesian coordinates (x^1, x^2, x^3), the Euclidean metric is a Riemannian metric defined by

$$\boldsymbol{g} = \delta_{ij}\,\boldsymbol{dx^i} \otimes \boldsymbol{dx^j}\,. \tag{39}$$

Such a metric is the metric the metric used in non-relativistic physics. Clearly in the presence of the Euclidean metric written in cartesian coordinates the distinction among vectors and 1-forms is immaterial. Of course this not true when different coordinates are adopted.

In special relativity one considers the 4-dimensional Minkowski space $\mathbb{R}^4$ endowed with Minkowski metric written in a global set of cartesian-like coordinates $x^0 = ct, x^1, x^2, x^2$, where t is the time coordinate of a point (event) in the Minkowski spacetime [2]

$$\boldsymbol{g} = -\boldsymbol{dx^0} \otimes \boldsymbol{dx^0} + \boldsymbol{dx^1} \otimes \boldsymbol{dx^1} + \boldsymbol{dx^2} \otimes \boldsymbol{dx^2} + \boldsymbol{dx^3} \otimes \boldsymbol{dx^3} \equiv \eta_{\mu\nu}\,\boldsymbol{dx^\mu} \otimes \boldsymbol{dx^\nu}\,; \tag{40}$$

where we have used the standard notation in special relativity according with greek indices run from 0 to 3.

In a space space endowed with a metric the distinction among vectors and 1-form is conventional. In physics where a a metric is almost always present indices are raised/lowered with a the metric tensor, namely the components of a vector u^μ can be turned into the components of a 1-form and viceversa according to

$$v_\mu = g_{\mu\nu}\,u^\mu\,, \qquad v^\mu = g^{\mu\nu}\,u_\nu\,. \tag{41}$$

The same is true for a tensor T^μ_ν

$$T_{\mu\nu} = T^\alpha_\nu\,g_{\alpha\mu}\,, \qquad T^{\mu\nu} = T^\mu_\alpha\,g^{\alpha\nu}\,. \tag{42}$$

[2] c is the speed of light such that ct has the dimension of a length.

In non-relativistic physics one deals with a space $\mathbb{R}^3$ endowed with an Euclidean metric given in cartesian coordinates by (39) while in special relativity the Minkowski metric (40) in $\mathbb{R}^4$ is used. In both cases the metric is a given non-dynamical object; things change radically in general relativity where the four dimension metric becomes a dynamical field determined by the matter content by Einstein field equations, a set of non-linear partial differential equations. Notice that in this case the metric field changes from point to point. For instance the famous Schwarzschild metric which describes a non-rotating black hole has the following form in spherical coordinates $(t,\, r,\, \theta,\, \varphi)$

$$\begin{aligned} &\boldsymbol{g} = -h\,\boldsymbol{dt}\otimes\boldsymbol{dt} + h^{-1}\,\boldsymbol{dr}\otimes\boldsymbol{dr} + r^2\left[\boldsymbol{d\theta}\otimes\boldsymbol{d\theta} + \sin(\theta)^2\,\boldsymbol{d\varphi}\otimes\boldsymbol{d\varphi}\right]\ ; \\ &h = 1 - \frac{2\,M\,G}{r}\ ; \end{aligned} \tag{43}$$

where M is the black hole mass and G is the Newton constant. It is clear that for large r, e.g. far form the black hole, the metric reduces to the Minkowski one in spherical coordinates.

When a Riemannian metric is given, one can naturally measures the length $L(\gamma)$ of a curve γ. Let $\boldsymbol{v}$ the tangent vector and P_a and P_b the initial and final points of γ; we have

$$L(\gamma) = \int_a^b \left[\boldsymbol{g}(\boldsymbol{v},\boldsymbol{v})\right]^{1/2} dt\,; \tag{44}$$

where $\gamma(t_a) = P_a$ and $\gamma(t_b) = P_b$.

7 Derivative of a Tensor?

In physical application in Euclidean spece one often needs to take the derivative of a tensor; take for instance the Maxwell equations for the electric and magnetic fields. Consider a 1-form field $\boldsymbol{\omega} \in T(U)^*$

$$\boldsymbol{\omega} = \omega_a \boldsymbol{dx}^a\,. \tag{45}$$

Can we form a $(0,2)$ of components $\Omega_{ab} = \partial_a\omega_b$ by taking the derivative of the components ω_b of $\boldsymbol{\omega}$ in a coordinates basis ? There a number of geometrical and physical reasons to forbid tha. For now let us fly very low and check whether the definition goes beyond a special system of coordinates.

The components of the would-be tensor under a coordinate transformation $x \to x'$ should trasform as

$$\Omega'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\Omega_{cd}\,; \tag{46}$$

on the other hand

$$\begin{aligned} &\frac{\partial}{\partial x'^a}\omega'_b = \frac{\partial}{\partial x'^a}\left(\frac{\partial x^n}{\partial x'^b}\omega_n\right) = \frac{\partial \omega_n}{\partial x'^a}\frac{\partial x^n}{\partial x'^b} + \frac{\partial^2 x^n}{\partial x'^a \partial x'^b}\omega_n = \frac{\partial \omega_n}{\partial x^c}\frac{\partial x^c}{\partial x'^a}\frac{\partial x^n}{\partial x'^b} \\ &= \Omega_{cn}\frac{\partial x^c}{\partial x'^a}\frac{\partial x^n}{\partial x'^b} + \omega_n \frac{\partial^2 x^n}{\partial x'^a \partial x'^b}\,; \end{aligned} \tag{47}$$

Thus unless

$$\frac{\partial^2 x^n}{\partial x'^a \partial x'^b} = 0 \tag{48}$$

the components of the would-be new tensor *does not* trasform properly and in general this will be not the case. As a result, in general, taking the partial derivative of a tensor does not produce a new tensor with one more slot. The bottom line is that to produce a new tensor by derivation an additional structure is needed. An exception is a *linear* coordinate transformation of the form

$$x^a = \Lambda^a_b\, x^b\,, \qquad \Lambda \text{ constant matrix}\,, \tag{49}$$

for which the partial derivative of the Jacobian identically vanish. This is the case for orthogonal transformation relating two set of cartesian coordinates whose axises are rotated or for the coordinate transformation relating two inertial observers in special relativity namely a Lorentz transformation. In this simplified scenario one can "forget about" the above mentioned difficulties which resurface when for instance spherical coordinates are considered.

8 Tensors in Euclidean $\mathbb{R}^3$

Take $\mathbb{R}^3$ with the Euclidean metric (39) and use global Cartesian coordinates. In the presence of a metric, given a vector, a 1-form can be naturally associated to it and viceversa. As discussed before, in Cartesian global coordinates with an Euclidean metric, the distinction among upper and lower indices is immaterial; nevertheless for pedagogical reasons we still stick to our general notation.

Tensor of inertia

The angular velocity of a rigid body $\boldsymbol{\omega}$ and its angular momentum $\boldsymbol{L}$ with respect of the center of mass can be obtained from the contraction of the vector $\boldsymbol{\omega}$ with the tensor of $\boldsymbol{I}$ of type $(1,1)$

$$\boldsymbol{L} = \boldsymbol{I}(\cdot, \boldsymbol{\omega})\,, \qquad \boldsymbol{I} = I^a_b\, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^a}} \otimes \boldsymbol{dx}^b\,; \tag{50}$$

with

$$I^a_b = \int_V d^3x\, \rho\, (\delta^a_b + x^a\, x^c\, \delta_{bc})\ ; \tag{51}$$

the integrale is performed in the volume V of the solid having mass density ρ. A further contraction gives twice the kinetic energy of the body in the center of mass

$$K = \frac{1}{2}\boldsymbol{I}(\boldsymbol{\omega}, \boldsymbol{\omega})\,. \tag{52}$$

Stress Tensor

Given a scalar quantity ρ carried by a current $\boldsymbol{J}$, the differential conservation law takes the form

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0\,. \tag{53}$$

This is the case for the electric charge, the mass for a non-relativistic fluid. A similar conservation low exists when the conserved quantity is not a scalar but a vector. For a continuous medium, the conservation of momentum takes the form

$$\partial_t \boldsymbol{G} + \boldsymbol{\nabla} \cdot \boldsymbol{T} = 0\,, \tag{54}$$

or in by using Cartesian coordinates

$$\partial_t G^a + \delta^{bc}\, \partial_b T^a_c\,; \tag{55}$$

where $\boldsymbol{G}$ is the momentum density and $\boldsymbol{T}$ is the stress tensor. In the case of the electromagnetic field

$$T_{ab} = \delta_{ab} \frac{(\boldsymbol{E}^2 + \boldsymbol{B}^2)}{8\,\pi} - \frac{E_a E_b + B_a B_b}{4\pi}\,. \tag{56}$$

The geometrical coordinate free form is

$$\boldsymbol{T} = \boldsymbol{g}\ \frac{[\boldsymbol{g}(\boldsymbol{E}, \boldsymbol{E}) + \boldsymbol{g}(\boldsymbol{B}, \boldsymbol{B})]}{8\pi} - \frac{1}{4\pi}\,(\boldsymbol{E} \otimes \boldsymbol{E} + \boldsymbol{B} \otimes \boldsymbol{B})\ ; \tag{57}$$

where $\boldsymbol{g}$ is the Euclidean metric.

8.1 Exercises

1. Find one more example in physics involving tensors.

2. Prove (32).

3. Derive eq. (28).

4. Find the the components of the Euclidean metric in spherical and cylindrical coordinates together with its inverse metric in the same coordinate systems.

5. Find the components of the Minkowski metric and the inverse Minkowski metric in spherical coordinates.

6. Given a generic vector in $\mathbb{R}^3$ endowed with the Euclidean metric, find its components in cylindrical and spherical coordinates and in the same coordinates find the associated 1-form by using the metric.

7. Choosing a local coordinates system write the explicit form of (44) and show the the length does not depend on the choice of parametrization of the curve.

9 Differential Forms

A special type of tensors of type $(0,p)$ play an important role. Start with a tensor $\boldsymbol{\Omega}$ of type $(0,2)$

$$\boldsymbol{\Omega} = \Omega_{ab}\, \boldsymbol{dx^a} \otimes \boldsymbol{dx^b}\,; \tag{58}$$

with the additional following symmetry condition

$$\boldsymbol{\Omega}(\boldsymbol{u},\boldsymbol{v}) = -\boldsymbol{\Omega}(\boldsymbol{v},\boldsymbol{u}) \qquad \boldsymbol{u},\boldsymbol{v} \in T(U); \tag{59}$$

or equivalently $\Omega_{ab} = -\Omega_{ba}$. Thanks to antisymmetry we can write $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = \frac{1}{2}\Omega_{ab}\, \left(\boldsymbol{dx}^a \otimes \boldsymbol{dx}^b - \boldsymbol{dx}^b \otimes \boldsymbol{dx}^a\right)\,. \tag{60}$$

Defining the exterior product among two 1-forms as

$$\boldsymbol{dx}^a \wedge \boldsymbol{dx}^b \equiv \boldsymbol{dx}^a \otimes \boldsymbol{dx}^b - \boldsymbol{dx}^b \otimes \boldsymbol{dx}^a\,; \tag{61}$$

thus The most general antisymmetric $(0,2)$ tensor, called 2-form, can be written as

$$\boldsymbol{\Omega} = \frac{1}{2}\Omega_{ab}\,\boldsymbol{dx}^a \wedge \boldsymbol{dx}^b\,. \tag{62}$$

In n dimension, the number of independent components is $n(n-1)/2$.
In the case of a $p-$form is a tensor of type $(0,p)$ completely antisymmetric, namely

$$\boldsymbol{\Omega}(\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_p) = \mathrm{Sign}(\sigma)\,\boldsymbol{\Omega}(\boldsymbol{u}_{\sigma(1)}, \boldsymbol{u}_{\sigma(2)}, \cdots, \boldsymbol{u}_{\sigma(p)})\,; \tag{63}$$

where $\mathrm{Sign}(\sigma)$ is the sign of the permutation σ that takes $1, 2, \cdots, p$ into $\sigma(1), \sigma(2), \cdots, \sigma(p)$. Thus,

$$\boldsymbol{\Omega} = \frac{1}{p!}\Omega_{a_1 a_2 \cdots a_p}\,\boldsymbol{dx}^{a_1} \wedge \boldsymbol{dx}^{a_2} \wedge \cdots \wedge \boldsymbol{dx}^{a_p}\,, \tag{64}$$

with

$$\boldsymbol{dx}^{a_1} \wedge \boldsymbol{dx}^{a_2} \wedge \cdots \wedge \boldsymbol{dx}^{a_p} = \sum_\sigma \mathrm{Sign}(\sigma)\,\boldsymbol{dx}^{a_1} \otimes \boldsymbol{dx}^{a_2} \otimes \cdots \otimes \boldsymbol{dx}^{a_p} \tag{65}$$

and the sum is taken over all $p!$ permutions of the group of p indices with each of then can take n possible values. Conventionally, a function on U is called a $0-$form.

Differential forms can be naturally multiplied my using the wedge (exterior product); taking a p-form $\boldsymbol{A}$ and a q-form $\boldsymbol{b}$

$$\boldsymbol{A} = \frac{1}{p!}A_{a_1 \cdots a_p}\boldsymbol{dx}^{a_1} \wedge \cdots \wedge \boldsymbol{dx}^{a_q}\,, \qquad \boldsymbol{B} = \frac{1}{q!}B_{b_1 \cdots b_q}\boldsymbol{dx}^{b_1} \wedge \cdots \wedge \boldsymbol{dx}^{p_q}\,, \tag{66}$$

then

$$\boldsymbol{A} \wedge \boldsymbol{B} = \frac{1}{q!p!}A_{a_1 \cdots a_p}B_{b_1 \cdots b_q}\,\boldsymbol{dx}^{a_1} \wedge \cdots \wedge \boldsymbol{dx}^{a_q} \wedge \boldsymbol{dx}^{b_1} \wedge \cdots \wedge \boldsymbol{dx}^{p_q} \tag{67}$$

is a $p+q$ form. Notice that

$$\boldsymbol{A} \wedge \boldsymbol{B} = (-1)^{p+q}\,\boldsymbol{B} \wedge \boldsymbol{A}\,. \tag{68}$$

Example
Consider a 2-dimensional space. The most general 1-form field is

$$\boldsymbol{\omega} = f_1(x1, x2)\boldsymbol{dx^1} + f_2(x1, x2)\boldsymbol{dx^2}\,. \tag{69}$$

The most general 2-form can be obtained by taking the product of two generic 1-forms

$$\begin{aligned}
&\boldsymbol{\omega} \wedge \boldsymbol{\omega}' = \left[f_1(x^1,x^2)\boldsymbol{dx^1} + f_2(x^1,x^2)\boldsymbol{dx^2}\right] \wedge \left[g_1(x^1,x^2)\boldsymbol{dx^1} + g_2(x^1,x^2)\boldsymbol{dx^2}\right] \\
&f_1 g_1\, \boldsymbol{dx^1} \wedge \boldsymbol{dx^1} + f_1 g_2\, \boldsymbol{dx^1} \wedge \boldsymbol{dx^2} + f_2 g_1\, \boldsymbol{dx^2} \wedge \boldsymbol{dx^1} + f_2 g_2\, \boldsymbol{dx^2} \wedge \boldsymbol{dx^2} \\
&= (f_1 g_2 - f_2 g_1)\, \boldsymbol{dx^1} \wedge \boldsymbol{dx^2} \equiv f(x^1,x^2)\boldsymbol{dx^1} \wedge \boldsymbol{dx^2}\,.
\end{aligned} \tag{70}$$

Thus, in the case $n = 2$, while in the linear space of 1-forms in a point is two-dimensional, the linear space of 2-forms is one-dimensional. Notice that, by antisymmetry the product of a generic 2-form with any form (expect 0-forms) gives zero; indeed

$$f\boldsymbol{dx^1} \wedge \boldsymbol{dx^2} \wedge \boldsymbol{A} = f f_1\, \boldsymbol{dx^1} \wedge \boldsymbol{dx^2} \wedge \boldsymbol{dx^1} + f f_2\, \boldsymbol{dx^1} \wedge \boldsymbol{dx^2} \wedge \boldsymbol{dx^2} = 0\,. \tag{71}$$

For $n = 2$ p-forms with $p > 2$ are trivial.

Example

Consider a 3-dimensional space. This time the most general 1-form field is

$$\boldsymbol{\omega} = f_1\boldsymbol{dx^1} + f_2\boldsymbol{dx^2} + f_3\boldsymbol{dx^3}\,. \tag{72}$$

The exterior product of two generic 1-forms reads

$$\begin{aligned}
&\boldsymbol{\omega} \wedge \boldsymbol{\omega}' = f_a\, \boldsymbol{dx^a} \wedge g_b\, \boldsymbol{dx^b} \\
&= (f_1 g_2 - f_2 g_1)\, \boldsymbol{dx^1} \wedge \boldsymbol{dx^2} + (f_1 g_3 - g_1 f_3)\, \boldsymbol{dx^1} \wedge \boldsymbol{dx^3} + (f_2 g_3 - g_2 f_3)\boldsymbol{dx^2} \wedge \boldsymbol{dx^3}\,.
\end{aligned} \tag{73}$$

For $n = 3$ the linear space of 2-forms with is 3-dimensional. The most general 3-form has the form

$$f\, \boldsymbol{dx^1} \wedge \boldsymbol{dx^2} \wedge \boldsymbol{dx^3}\,; \tag{74}$$

thus the linear of 3-forms with is 1-dimensional. For $n = 3$ p-forms with $p > 3$ are trivial.

The space of p-forms in U n-dimensional space is denoted by $\Lambda(U)^p$ and can can easily show that

$$\mathrm{Dim}[\Lambda(U)^p] = \binom{n}{p} = \frac{n!}{(n-p)!\, p!}\,. \tag{75}$$

It follows that in dimension $\Lambda(U)^p$ has the same dimension of $\Lambda(U)^{n-p}$. When a metric is present, we will see that there is a canonical way to map $\Lambda(U)^{n-p}$ into $\Lambda(U)^p$ and viceversa.

9.1 Exterior Derivative

As discussed in section 7, the partial derivative of a tensor is not a tensor. However in the case of differential forms, the antisymmetric built-in structure allows to get rid of the spurious term proportional to the derivative of the Jacobian the spoils the transformation properties. Given a p-form $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \frac{1}{p!}\omega_{a_1\cdots a_p}\, \boldsymbol{dx}^{a_1} \wedge \cdots \wedge \boldsymbol{dx}^{a_p} \,, \tag{76}$$

one define a p+1 $\boldsymbol{d\omega}$ form called the exterior derivative of $\boldsymbol{\omega}$ by

$$\boldsymbol{d\omega} = \frac{1}{p!}\partial_b\omega_{a_1\cdots a_p}\, \boldsymbol{dx}^{b} \wedge \boldsymbol{dx}^{a_1} \wedge \cdots \wedge \boldsymbol{dx}^{a_p} \,. \tag{77}$$

One can easily verify that from the definition

$$\boldsymbol{d}\cdot\boldsymbol{d} = \boldsymbol{d^2} = 0\,. \tag{78}$$

Example
Take a function f (0-form) then

$$\boldsymbol{d}f = \partial_a f \boldsymbol{dx}^a\,. \tag{79}$$

Taking its exterior derivative

$$\boldsymbol{d^2}f = \boldsymbol{d}\left(\partial_a f \boldsymbol{dx}^a\right) = \partial_a\partial_b f\, \boldsymbol{dx}^a \wedge \boldsymbol{dx}^b = 0\,. \tag{80}$$

Example
Take $n = 3$ and a generic 1-form

$$\boldsymbol{\omega} = \omega_a\, \boldsymbol{dx}^a\,; \tag{81}$$

then

$$\begin{aligned}
&\boldsymbol{d\omega} = \partial_b\omega_a\, \boldsymbol{dx}^b \wedge \boldsymbol{dx}^a \\
&= \partial_1\omega_2\, \boldsymbol{dx}^1 \wedge \boldsymbol{dx}^2 + \partial_1\omega_3\, \boldsymbol{dx}^1 \wedge \boldsymbol{dx}^3 + \partial_2\omega_1\, \boldsymbol{dx}^2 \wedge \boldsymbol{dx}^1 + \partial_2\omega_3\, \boldsymbol{dx}^2 \wedge \boldsymbol{dx}^3 \\
&+ \partial_3\omega_1\, \boldsymbol{dx}^3 \wedge \boldsymbol{dx}^1 + \partial_3\omega_2\, \boldsymbol{dx}^3 \wedge \boldsymbol{dx}^2 \\
&= \left(\partial_1\omega_2 - \partial_2\omega_1\right)\, \boldsymbol{dx}^1 \wedge \boldsymbol{dx}^2 + \left(\partial_1\omega_3 - \partial_3\omega_1\right)\, \boldsymbol{dx}^1 \wedge \boldsymbol{dx}^3 \\
&+ \left(\partial_2\omega_3 - \partial_3\omega_2\right)\, \boldsymbol{dx}^2 \wedge \boldsymbol{dx}^3\,.
\end{aligned} \tag{82}$$

The components of $\boldsymbol{d\omega}$ are the same of the curl of vector with the same components of $\boldsymbol{\omega}$.
As mentioned befor the antisymmetry of forms fixes the issue of the transformation property. Indeed, under a coordinate transformation $x \to x'(x)$ one has

$$\begin{aligned}
\boldsymbol{d\omega} &= \partial'_k \omega'_j \, \boldsymbol{dx'^k} \wedge \boldsymbol{dx'^j} = \partial'_k \left(\frac{\partial x^i}{\partial x'^j} \, \omega_i \right) \boldsymbol{dx'^k} \wedge \boldsymbol{dx'^j} \\
&= \left[\frac{\partial x^i}{\partial x'^j} \frac{\partial \omega_i}{\partial x'^k} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k} \, \omega_i \right] \boldsymbol{dx'^k} \wedge \boldsymbol{dx'^j} = \frac{\partial x^i}{\partial x'^j} \frac{\partial \omega_i}{\partial x'^k} \, \boldsymbol{dx'^k} \wedge \boldsymbol{dx'^j} \\
&= \frac{\partial \omega_i}{\partial x^k} \, \boldsymbol{dx}^k \wedge \boldsymbol{dx}^i \, ;
\end{aligned} \tag{83}$$

the terms proportional to $\partial^2 x^i / \partial x'^j \partial x'^k$ vanishes by antisymmetry and we have used that $\boldsymbol{d}x^i = \frac{\partial x^i}{\partial x'^j} \boldsymbol{dx'^j}$.

Finally notice that given a vector $\boldsymbol{\xi}$ and a function f we can introduce the 1-form $\boldsymbol{df}$ by

$$\boldsymbol{\xi}(f) \equiv \boldsymbol{df}(\boldsymbol{\xi}) = \xi^i \, \partial_i f \tag{84}$$

thus

$$\boldsymbol{df} = \partial_i f \, \boldsymbol{dx^i} \, ; \tag{85}$$

The differential of a function f is naturally a 1-form. Thus, intuitively a 1-form can be interpreted as the infinitesimal variation of some quantity in a unspecified direction; only when a vector field is provided, say $\boldsymbol{\xi}$ we get a scalar quantity given the same variation along the direction associated with the vector. For instance the infinitesimal work done by a force is a naturally a 1-form which, when contracted with the vector tangent to curve describing the motion of a particle, gives a scalar quantity that can be integrated along the curve to obtain the total work done.

Differential forms are also crucial in potential theory and differential equations. Consider the problem of a p-form $\boldsymbol{\omega}$ and ask the question it exists a p-1 form $\boldsymbol{\Omega}$ such that

$$\boldsymbol{d\Omega} = \boldsymbol{\omega} \, . \tag{86}$$

When such a form exists, the form $\boldsymbol{\omega}$ is called exact. Taking the exterior derivative of the above relation on get immediately the following necessary condition

$$\boldsymbol{d\omega} = 0 \, . \tag{87}$$

A form with vanishing exterior derivative is called closed. In general manifold a closed form can be closed but not exact. However, locally, a closed form is also exact thanks to the Poincaré lemma.

Poincaré Lemma

In open region region of $\mathbb{R}^n$ a closed form is also locally exact.

Example

Take $\mathbb{R}^2/(0,0)$, i.e.. the origin is removed, and consider the following 1-form

$$\boldsymbol{\omega} = \frac{x\,\boldsymbol{dy} - y\,\boldsymbol{dx}}{x^2+y^2}\,. \tag{88}$$

One has

$$\begin{aligned} \boldsymbol{d\omega} &= \frac{2\boldsymbol{dx}\wedge\boldsymbol{dy}}{x^2+y^2} + (-1)^2\frac{(x\boldsymbol{dy} - y\boldsymbol{dx})\wedge\boldsymbol{d}(x^2+y^2)}{(x^2+y^2)^2} \\ &\frac{2\boldsymbol{dx}\wedge\boldsymbol{dy}}{x^2+y^2} + 2\frac{(x^2+y^2)\,\boldsymbol{dy}\wedge\boldsymbol{dx}}{(x^2+y^2)^2} = 0\,. \end{aligned} \tag{89}$$

The form is closed and by the Poincaré lemma is locally exact. Indeed

$$\boldsymbol{\omega} = \boldsymbol{d}f\,, \qquad f(x,y) = \arctan\left(\frac{y}{x}\right)\,. \tag{90}$$

Notice that f is not defined in the origin and is not single valued, being $f=\theta$ the polar angle in polar coordinates. Thus, f is not globally defined and then $\boldsymbol{\omega}$ exists only locally.

9.2 Integration and the Stokes Theorem

9.3 Hodge Dual

In The presence of a metric the fact that

$$\mathrm{Dim}\left(\Lambda(U)^{n-p}\right) = \mathrm{Dim}\left(\Lambda(U)^p\right) \tag{91}$$

can be used to build a canonical map. In a n-dimensional manifold, given a local coordinate basis, the most general n-forms (volume form) can be written as

$$\Omega = f\,\boldsymbol{dx}^1\wedge\boldsymbol{dx}^2\wedge\cdots\wedge\boldsymbol{dx}^n\,, \tag{92}$$

where f is a function. Defining the Levi-Civita symbol

$$\epsilon_{i_1 i_2 \cdots i_n} = \begin{cases} 1 & i_1 = 1,\, i_2 = 1,\, \cdots, i_n = n \\ 0 & \text{if any of the indices have the same value} \\ -1 & \text{odd permutation of } i_1 = 1,\, i_2 = 1,\, \cdots, i_n = n \\ 1 & \text{even permutation of } i_1 = 1,\, i_2 = 1,\, \cdots, i_n = n \end{cases} \tag{93}$$

For instance when $n = 4$

$$\epsilon_{2134} = \epsilon_{1243} = -1\,, \qquad \epsilon_{4123} = 1\,, \qquad \epsilon_{1134} = 0\,. \tag{94}$$

By using the Levi-Civita symbol $\boldsymbol{\Omega}$ can be rewritten as

$$\Omega = \frac{f}{n!}\, \epsilon_{i_1 i_2 \cdots i_n}\, \boldsymbol{dx}^{i_1} \wedge \boldsymbol{dx}^{i_2} \wedge \cdots \wedge \boldsymbol{dx}^{i_n}\,. \tag{95}$$

Let us now change coordinates $x \to x'$; in the new coordinates the same form is written as

$$\begin{aligned} \Omega &= \frac{f}{n!} \frac{\partial x^{i_1}}{\partial x'^{j_1}} \frac{\partial x^{i_2}}{\partial x'^{j_2}} \cdots \frac{\partial x^{i_n}}{\partial x'^{j_n}}\, \epsilon_{i_i i_2 \cdots i_n}\, \boldsymbol{dx'}^{j_1} \wedge \boldsymbol{dx'}^{j_2} \wedge \cdots \wedge \boldsymbol{dx'}^{j_n} \\ &= \frac{f}{n!} \operatorname{Det}(\boldsymbol{J}^{-1})\, \epsilon'_{j_1 j_2 \cdots j_n}\, \boldsymbol{dx'}^{j_1} \wedge \boldsymbol{dx'}^{j_2} \wedge \cdots \wedge \boldsymbol{dx'}^{j_n}\,; \end{aligned} \tag{96}$$

where the matrix $\boldsymbol{J}$ is the Jacobian of the transformation: $J^a_b = \partial x'^a / \partial x^b$. Thus, we infer that

$$\epsilon'_{i_1 i_2 \cdots i_n} = \operatorname{Det}(\boldsymbol{J})\, \epsilon_{i_1 i_2\,; \cdots i_n} \tag{97}$$

we know notice that from the transformation properties of the metric

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b}\, g_{cd} \tag{98}$$

we get

$$\operatorname{Det}(\boldsymbol{g'}) = \operatorname{Det}(\boldsymbol{J})^{-2} \operatorname{Det}(\boldsymbol{g})\,. \tag{99}$$

Setting

$$g = |\operatorname{Det}(g)|\,; \tag{100}$$

then

$$\frac{\sqrt{g'}}{\sqrt{g}} = |\operatorname{Det}(\boldsymbol{J})|^{-1}\,. \tag{101}$$

As result for coordinate transformation for which $\mathrm{Det}(\boldsymbol{J}) > 0$, e.g. orientation preserving [3], (97) can written as

$$\sqrt{g'}\,\epsilon'_{i_1 i_2 \cdots i_n} = \sqrt{g}\,\epsilon_{i_1 i_2 \cdots i_n}\,; \tag{102}$$

namely: the Levi-Civita symbol times $\sqrt{g}$ under an orientation preserving coordinate transformation is an invariant tensor; in the following we will the following definition

$$E_{i_1 i_2 \cdots i_n} = \sqrt{g}\,\epsilon_{i_1 i_2 \cdots i_n}\,. \tag{103}$$

From the definition it is straightforward to derive some useful properties of E. Raising with metric all the indices we have

$$E^{i_1 i_2 \cdots i_n} = \frac{1}{\sqrt{g}}\,\epsilon^{i_1 i_2 \cdots i_n}\,; \tag{104}$$

the Levi-Civita symbol $\epsilon^{i_1 i_2 \cdots i_n}$ has the very same definition (93) of the one with covariant indices. We have

$$E^{i_1 i_2 \cdots i_n}\,E_{i_1 i_2 \cdots i_n} = n!\,; \tag{105}$$

setting $n = 3$

$$E^{ijk}\,E_{imn} = \left(\delta^j_m\,\delta^k_n - \delta^j_n\,\delta^k_m\right)\,; \tag{106}$$

while for $n = 4$

$$E^{ijkl}\,E_{ijmn} = 2\left(\delta^k_m\,\delta^l_n - \delta^k_n\,\delta^l_m\right)\,. \tag{107}$$

Similar relations can be derived by noticing that the product of two Es is just the antisymmetric product of Kronecker deltas.

The freshly defined invariant tensor can be used to the define the Hodge dual. Given a p-form $\boldsymbol{\omega}$, a n-p-form $*\boldsymbol{\omega}$ is defined by

$$*\boldsymbol{\omega} = \frac{\sqrt{g}\,\epsilon_{j_1 j_2 \cdots j_{n-p} i_1 i_2 \cdots i_p \cdots i_n}}{p!(n-p)!}\,\omega^{i_1 i_2 \cdots i_p}\,\boldsymbol{dx}^{j_1} \wedge \boldsymbol{dx}^{j_2} \wedge \cdots \wedge \boldsymbol{dx}^{j_{n-p}}\,; \tag{108}$$

where

$$\omega^{i_1 i_2 \cdots i_p} = g^{i_1 k_1}\,\cdots g^{i_p k_k}\,\omega_{k_1 \cdots k_p}\,. \tag{109}$$

Applying twice the Hodge map one gets the identity, modulo some numerical constants,

$$**\boldsymbol{\omega} = \mathrm{sgn}\left[\mathrm{Det}(\boldsymbol{g})\right]\,(-1)^{p(n-p)}\,\boldsymbol{\omega}\,. \tag{110}$$

[3]Notice that while the original transformation properties of the Levi-Civita symbol depends on the orientation, the sign of $\mathrm{Det}(\boldsymbol{J})$, this is not the case for (101).

In the definition of the Hodge dual the metric is needed in the definition of the invariant symbol E and to rise the components of the original form.

Curl of vector

Consider $n = 3$ and and a manifold with metric. Take a vector $\boldsymbol{V}$, by using the Hodge dual one define in a covariant way the notion of $\boldsymbol{\nabla} \wedge \boldsymbol{V}$. From V one can define the 1-form

$$\widetilde{\boldsymbol{V}} = \boldsymbol{g}(\boldsymbol{V}) = g_{ab} V^a \, \boldsymbol{dx}^a \,. \tag{111}$$

As already discussed $\boldsymbol{d}\widetilde{\boldsymbol{V}}$ is a 2-form whose components reassembles the one the curl. The dual of a 2-form is a 1-form that can be turned back into a vector by using the inverse metric $\tilde{\boldsymbol{g}}$, thus we define

$$\boldsymbol{\nabla} \wedge \boldsymbol{V} = \tilde{\boldsymbol{g}}(*\boldsymbol{g}(\boldsymbol{V}))\,. \tag{112}$$

By setting $V_a = g_{ab} V^a$, we have

$$\boldsymbol{\nabla} \wedge \boldsymbol{V} = g^{ab} \, E_{bcd} \, g^{cm} \, g^{dn} \, (\partial_m V_n - \partial_m V_n) \, \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}^a} \,. \tag{113}$$

The above expression can be used to find the curl of vector in the Euclidean space in any coordinate system.

By using the Hodge dual one can also define an operato that, differently from d, lowers the rank of a form. Indeed applying $*d*$ to a p-form one gets a p-1-form. Defining the coderivative $\boldsymbol{\delta}$ on a p-form by

$$\boldsymbol{\delta} = (-1)^{n+1+p\,n} \; * \boldsymbol{d} * ; \tag{114}$$

notice that from its definition $\boldsymbol{\delta}^2 = 0$. It is possible define the generalization of the Laplacian operator in a generic manifold by

$$\Delta = (\boldsymbol{d} + \boldsymbol{\delta})^2 = \boldsymbol{d}\,\boldsymbol{\delta} + \boldsymbol{\delta}\,\boldsymbol{d}\,. \tag{115}$$

Applying Δ to a 0-from f one gets in a generic coordinate system

$$\Delta f = \frac{\mathrm{sgn}\,[\mathrm{Det}(\boldsymbol{g})]}{\sqrt{g}} \partial_k \left(\sqrt{g} \, g^{kj} \, \partial_j f \right) \,. \tag{116}$$

By using the exterior derivative, the coderivative and Δ one can generalized the harmonic analysis in $\mathbb{R}^3$ to a generic manifold with a metric.

9.4 Thermodynamics

A simple thermodynamical system can be described in terms of pressure P, volume V and temperature with two independent variables (fixed number of particles). Thus, we are dealing a subset of $\mathbb{R}^2$. The first principle can be written as a relation among 1-forms involving

- the work 1-form $\boldsymbol{\lambda} = -P\boldsymbol{d}V$;
- heat 1-form $\boldsymbol{q}$;
- the state function internal energy U.

The first principle reads

$$\boldsymbol{d}U = \boldsymbol{q} + \boldsymbol{\lambda}\,. \tag{117}$$

taking the exterior derivative of the first principle we get

$$\boldsymbol{d}\,(\boldsymbol{q} + \boldsymbol{\lambda}) = 0\,. \tag{118}$$

While $\boldsymbol{q} + \boldsymbol{\lambda}$ is closed, nor $\boldsymbol{\lambda}R$ neither $\boldsymbol{q}$ are closed. Consider now a perfect gas with the equation of state and internal energy

$$P\,V = n\,R\,T\,, \qquad U = n\,c_v\,T\,. \tag{119}$$

Let us derive the Mayer relation. Let us use independent variables (coordinates) (T, P), the volume is a function of T and P that can be obtained from (119). Now

$$\boldsymbol{q} = A\,\boldsymbol{dT} + B\,\boldsymbol{d}P \equiv n\,c_P\,\boldsymbol{d}T + B\,\boldsymbol{d}P\,; \tag{120}$$

where by definition A is the heat capacity at constant pressure. The first principle can be written as

$$\begin{aligned} n\,c_V\,\boldsymbol{d}T &= n\,c_P\,\boldsymbol{d}T + B\,\boldsymbol{d}P - P\,\boldsymbol{d}\left(\frac{n\,R\,T}{P}\right) \\ &\Rightarrow (n\,c_V - n\,c_P - n\,R)\,\boldsymbol{d}T + \left(B - \frac{n\,R\,T}{P}\right)\boldsymbol{d}P = 0 \\ &\Rightarrow c_P = R + c_V\,, \qquad B = \frac{n\,R\,T}{P}\,; \end{aligned} \tag{121}$$

the first relation is the Mayer's relation, the second gives B and then

$$\boldsymbol{q} = n\,c_P\,\boldsymbol{d}T + \frac{n\,R\,T}{P}\,\boldsymbol{d}P\,. \tag{122}$$

Though $\boldsymbol{q}$ and $\boldsymbol{\lambda}$ are not exact, the relation (118) guarantees that there are two functions f and S such that

$$\boldsymbol{q} = f\,\boldsymbol{d}S\,. \tag{123}$$

Indeed, for a perfect gas it easy to find f and S. Consider as independent variables T and V; the heat 1-form is given by

$$\boldsymbol{q} = n\,c_V\,\boldsymbol{d}T + \frac{n\,R\,T}{V}\,\boldsymbol{d}V\,. \tag{124}$$

Dividing by T

$$\frac{\boldsymbol{q}}{T} = n\,c_V\,\frac{\boldsymbol{d}T}{T} + \frac{n\,R}{V}\,\boldsymbol{d}V = \boldsymbol{d}\left[n\,c_V\,\log(T) + n\,R\,\log(V)\right]\,. \tag{125}$$

Thus, by suitable choice of the integration constant, one gets the entropy of a perfect gas

$$S = n\left[c_V\,\log\left(\frac{T}{T_0}\right) + R\,\log\left(\frac{V}{V_0}\right)\right]\,, \tag{126}$$

and

$$\boldsymbol{q} = T\,\boldsymbol{d}S\,. \tag{127}$$

Maxwell relations are easy to derive by using differential forms. The first principle can be written as

$$\boldsymbol{d}U = T\,\boldsymbol{d}S - P\boldsymbol{d}V\,; \tag{128}$$

taking the exterior derivative

$$\boldsymbol{d}T\wedge\boldsymbol{d}S - \boldsymbol{d}P\wedge\boldsymbol{d}V = 0\,. \tag{129}$$

Taking as fundamental variables (S, V), both T and P should considered function of such fundamental variables; thus (129) gives

$$\begin{aligned}
&\left(\frac{\partial T}{\partial V}\right)_S \boldsymbol{d}V\wedge\boldsymbol{d}S + \frac{\partial T}{\partial S}_P\boldsymbol{d}S\wedge\boldsymbol{d}S - \left[\left(\frac{\partial P}{\partial V}\right)_S\boldsymbol{d}V + \left(\frac{\partial P}{\partial S}\right)_P\boldsymbol{d}S\right]\wedge\boldsymbol{d}V\\
&\Rightarrow \left[\left(\frac{\partial T}{\partial V}\right)_S + \left(\frac{\partial P}{\partial S}\right)_P\right]\boldsymbol{d}V\wedge\boldsymbol{d}S = 0\,;
\end{aligned} \tag{130}$$

as a result

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_P . \tag{131}$$

If we take as independent variables (coordinates) (T, V), (129) reads

$$\boldsymbol{d}T \wedge \boldsymbol{d}V \left(\frac{\partial S}{\partial V}\right)_T - \left(\frac{\partial P}{\partial T}\right)_V \boldsymbol{d}T \wedge \boldsymbol{d}V = 0\,; \tag{132}$$

which gives

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V . \tag{133}$$

The simple derivation of (131) and (133) gives a taste on how Maxwell relation can be easily derived simply

9.5 Covariant Formulation of Electrodynamics

Electrodynamics was crucial in the formulation of special relativity. Maxwell equations are not covariant under Galilean transformations and experimentally the speed of light is the same in all intertial systems. Special relativity can be formulated as the dynamics on a four-dimensional manifold equipped with the Minkowski metric (40)

$$\boldsymbol{g} = \eta_{\mu\nu}\, \boldsymbol{dx^\mu} \otimes \boldsymbol{dx^\nu}\,; \tag{134}$$

such a manifold is isomorphic to $\mathbb{R}^4$ and $x^\mu = (ct, x^i)$ are global coordinates. The class of coordinates linear transformations

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu \qquad \text{such that } \Lambda^\mu_\alpha\, \Lambda^\nu_\beta\, \eta_{\mu\nu} = \eta_{\alpha\beta} \tag{135}$$

are called Lorentz transformations and relate the coordinates of inertial observers using their clocks and spatial Cartesian coordinates; form small relative velocities $v/c << 1$ one gets back to Galilean transformations.

Let start from the Maxwell equations

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} + \tfrac{1}{c}\tfrac{\partial \boldsymbol{B}}{\partial t} = 0\,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{136}$$

$$\boldsymbol{\nabla} \wedge \boldsymbol{B} = \tfrac{1}{c}\tfrac{\partial \boldsymbol{E}}{\partial t} + \tfrac{4\pi}{c}\boldsymbol{J}\,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\,\rho\,. \tag{137}$$

Consistency requires the charge conservation expressed in a differential form as

$$\partial_t \rho + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0\,. \tag{138}$$

It is worth to tress that the Maxwell equations are *linear* equations for the fields. The first two equations allows locally to express the field in terms of a scalar and a vector potential; indeed from (136)

$$
\begin{aligned}
&\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \quad \Rightarrow \quad \exists\boldsymbol{A}: \quad \boldsymbol{B}=\boldsymbol{\nabla}\wedge\boldsymbol{A}\,;\\
&\boldsymbol{\nabla}\wedge\boldsymbol{E}+\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}=\boldsymbol{\nabla}\wedge\left(\boldsymbol{E}+\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\right)=0\\
&\Rightarrow\exists\varphi: \quad \boldsymbol{E}=-\boldsymbol{\nabla}\varphi-\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t}\,.
\end{aligned}
\tag{139}
$$

The remaining two Maxwell equations can be written as (see the exercise at the end of this section)

$$
\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}+\tfrac{1}{c}\tfrac{\partial\varphi}{\partial t}\right)-\Delta\boldsymbol{A}+\tfrac{1}{c^2}\tfrac{\partial^2\boldsymbol{A}}{\partial t^2}=\tfrac{4\pi}{c}\boldsymbol{J}\,;
\tag{140}
$$

$$
\Delta\varphi+\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\boldsymbol{A}=-4\pi\,\rho\,.
\tag{141}
$$

The formulation of the Maxwell equations in terms of the scalar and vector potentials introduce some redundancy, in the sense that there many potentials that produce the same the same observables electric and magnetic fields. Indeed, if we change $\boldsymbol{A}$ by a gradient of a function Λ, $\boldsymbol{B}$ does not change, similar consideration can be made for φ. The bottom line is that if we change the potentials according with

$$
\begin{aligned}
&\boldsymbol{A}\to\boldsymbol{A}'=\boldsymbol{A}-\boldsymbol{\nabla}\Lambda\,, \qquad \varphi\to\varphi'=\varphi-\frac{1}{c}\frac{\partial\Lambda}{\partial t}\\
&\boldsymbol{E}\to\boldsymbol{E}'\,, \qquad \boldsymbol{B}\to\boldsymbol{B}'\,,
\end{aligned}
\tag{142}
$$

the fields do not change. The transformation (142) is called gauge invariance. Theories that exhibits such an invariance are called gauge theories. As a matter of fact all known fundamental interactions: electromagnetism, weak interactions, strong interactions and gravity are gauge theories. In particular electrodynamics is a $U(1)$ gauge theory, where $U(1)$ refers to the type of gauge transformations. One may wonder why one should introduce the potentials instead to stick to the physically observable fields. There are to basic reasons. Often, it is easier to solve the Maxwell equations written in terms of the potentials. A more fundamental reasons is that there is no known way to quantize a system involving electromagnetic interactions relying only on the magnetic and electric fields; indeed quantization involves a Lagrangian and/or Hamiltonian formulation which requires the potentials. For instance, the Hamiltonian of a particle with charge q in the presence of a magnetic field is obtained by the following replacement rule for the canonical momentum

$$\boldsymbol{p} \to \boldsymbol{p} + \frac{q}{c}\boldsymbol{A}\,. \tag{143}$$

By exploiting gauge invariance (142), (140) and (141) can be decoupled. By choosing Λ such that

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0\,, \tag{144}$$

such a choice is called Lorentz gauge. In the Lorentz gauge (142), (140) take the very simple form

$$\Box \boldsymbol{A} = \tfrac{4\pi}{c}\,; \tag{145}$$

$$\Box \varphi = 4\pi\,\rho\,; \tag{146}$$

where we have introduce the d'Alembert wave operator

$$\Box = -\Delta + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,. \tag{147}$$

Let us now reformulate the Maxwell equations in manifest Lorentz in a geometric form. The scalar and vector potential can put together to form a 1-form of to be more precise, a connection [4]

$$\boldsymbol{A} = \varphi\,\boldsymbol{dx}^0 - A^1\,\boldsymbol{dx}^1 - A^2\,\boldsymbol{dx}^2 - A^3\,\boldsymbol{dx}^3 = A_\mu\,\boldsymbol{dx}^\mu\,. \tag{148}$$

Under a gauge transformation, see (142), we have that

$$\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{d}\Lambda \qquad \Rightarrow \qquad A_\mu \to {A'}_\mu = A_\mu + \partial_\mu \Lambda\,. \tag{149}$$

It is easy to guess how to express $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of the connection $\boldsymbol{A}$. The formers contains 6 independent gauge invariants components related to

[4] Notice the popular signature $(+,-,-,-)$ introduces some level of awkwardness to an ambiguity to associate a 3-vector with the upper or lower indices; some care is needed. With the signature $(-,+,+,+)$ that corresponds to an overall sign change in $\eta_{\mu\nu}$ such ambiguity is not present.

the potentials by a single derivative. The only object that can do the job is the following curvature 2-form

$$\boldsymbol{F} = \boldsymbol{dA} = \frac{1}{2} F_{\mu\nu}\, \boldsymbol{dx}^{\mu} \wedge \boldsymbol{dx}^{\nu} \,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{150}$$

One can check that

$$F_{0i} = E^i \quad i = 1, 2, 3 \,, \qquad F_{ij} \partial_i A_j - \partial_j = -\epsilon_{ijk}\, B^k \,. \tag{151}$$

Thus

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & -B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & B^2 & B^1 & 0 \end{pmatrix} . \tag{152}$$

The remaining two Maxwell equations can be also written geometrically by introducing the current 1-form

$$\boldsymbol{J} = c\rho\, \boldsymbol{dx}^0 - J^1\, \boldsymbol{dx}^1 - J^2\, \boldsymbol{dx}^2 - J^3\, \boldsymbol{dx}^3 = J_\mu\, \boldsymbol{dx}^\mu \,, \tag{153}$$

as $\boldsymbol{d} * \boldsymbol{F} = 4\pi/c * \boldsymbol{J}$. Thus Maxwell equation can be cast in geometric form as

$$\boldsymbol{dF} = 0 \,, \qquad \boldsymbol{d} * \boldsymbol{F} = \frac{4\pi}{c} * \boldsymbol{J} \,. \tag{154}$$

Notice that by the Poincaré lemma $\boldsymbol{dF} = 0$ is locally equivalent to $\boldsymbol{F} = \boldsymbol{dA}$. In Cartesian coordinates, the two Maxwell equations containing the sources assume the form

$$\partial^\nu F_{\mu\nu} = -\frac{4\pi}{c} J_\mu \,; \tag{155}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu$. Taking a second derivative

$$\partial^\mu \partial^\nu F_{\mu\nu} = -\frac{4\pi}{c} \partial^\mu J_\mu \;\Rightarrow \partial^\mu J_\mu = 0 \,; \tag{156}$$

the last equation is the charge conservation (138). By using the expression of the curvature tensor, we get

$$\partial_\mu \left(\partial^\nu A_\nu \right) - \Box A_\mu = -\frac{4\pi}{c} J_\mu \,; \tag{157}$$

notice that $\Box = \partial^\mu \partial_\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu$. In the Lorentz gauge (144) that can be written in compact form as $\partial^\mu A_\mu = 0$, we get

$$\Box A_\mu = \frac{4\pi}{c} J_\mu \,. \tag{158}$$

Notice that, while (160) is valid independently from the choice of coordinates, equations (156-158) are valid only Cartesian-like coordinates in which the Minkowski metric assume the familiar form

$$\eta_{mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{159}$$

We end by mentioning the electric magnetic duality. Maxwell equations at first sight seems to asymmetric in $\boldsymbol{E}$ and $\boldsymbol{B}$. Actually this is the case only because it has been assumed the there is no magnetic charge but only electric ones. Introdung the 1-form $\boldsymbol{J}_m$ magnetic current, the modified Maxwell equations reads

$$\boldsymbol{dF} = \frac{4\pi}{c} * \boldsymbol{J}_m \,, \qquad \boldsymbol{d} * \boldsymbol{F} = \frac{4\pi}{c} * \boldsymbol{J} \,. \tag{160}$$

In the new form the equation are invariant under the magnetic electric duality

$$\boldsymbol{F} \to *\boldsymbol{F} \,, \qquad \boldsymbol{J} \to \boldsymbol{J}_m \,. \tag{161}$$

9.6 Hamiltonian Mechanics

9.7 Exercises

1. By using (113) find in Euclidean space the expresion of the curl of a vector field in spherical coordinates.

2. Derive (116).

3. By using the properties of the Levi-Civita symbol in three dimensions, show that in Cartesian coordinates

$$\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{A}) = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \Delta \boldsymbol{A} \,.$$

4. Find the equation that the gauge transformation parameter Λ should satisfy in order that the gauge Lorentz condition (144) is satisfied.

10 Connection and Curvature